PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

A Theory of the Origin of Microseisms

M. S. Longuet-Higgins

Phil. Trans. R. Soc. Lond. A 1950 **243**, 1-35
doi: 10.1098/rsta.1950.0012

A THEORY OF THE ORIGIN OF MICROSEISMS

By M. S. LONGUET-HIGGINS

Department of Geodesy and Geophysics, University of Cambridge

(*Communicated by H. Jeffreys, F.R.S.—Received* 19 *September* 1949—
Revised 18 *March* 1950—*Read* 30 *March* 1950)

CONTENTS

In the past it has been considered unlikely that ocean waves are capable of generating microseismic oscillations of the sea bed over areas of deep water, since the decrease of the pressure variations with depth is exponential, according to the first-order theory generally used. However, it was recently shown by Miche that in the second approximation to the standing wave there is a second-order pressure variation which is not attenuated with depth and which must therefore ultimately predominate over the first-order pressure variations. In §§ 2 and 3 of the present paper the general conditions under which second-order pressure variations of this latter type will occur are considered. It is shown that in an infinite wave train there is in general a second-order pressure variation at infinite depth which is applied equally over the whole fluid and is associated with no particle motion. In the case of two progressive waves of the same wave-length travelling in opposite directions this pressure variation is proportional to the product of the (first-order) amplitudes of the two waves and is of twice their frequency. The pressure variation at infinite depth is found to be closely related to changes in the potential energy of the wave train as a whole. By introducing the two-dimensional frequency spectrum of the motion it is shown that in the general case variations in the mean pressure over a wide area only occur when the spectrum contains wave groups of the same wave-length travelling in opposite directions. (These are called opposite wave groups.)

In § 4 the effect of the compressibility of the water is considered by evaluating the motion of an opposite pair of waves in a heavy compressible fluid to the second order of approximation. In place of the pressure variation at infinite depth, waves of compression are set up, and there is resonance between the bottom and the free surface when the depth of water is about $(\frac{1}{2}n+\frac{1}{4})$ times the length of a compression wave (n being an integer). The motion in a surface layer whose thickness is of the order of the length of a Stokes wave is otherwise unaffected by the compressibility.

Section 5 is devoted to the question whether the second-order pressure variations in surface waves are capable of generating microseisms of the observed order of magnitude. By considering the displacement of the sea bed due to a concentrated force at the upper surface of the water it is shown that the effect of resonance will be to increase the disturbance by a factor of the order of 5 over its value in shallow water. The results of §§ 3 and 4 are used to derive an expression for the vertical displacement of the ground in terms of the frequency characteristics of the waves. The displacement from a storm area of 1000 sq.km. is estimated to be of the order of $6{\cdot}5\mu$, at a distance of 2000 km.

Ocean waves may therefore be the cause of microseisms, provided that there is interference between groups of waves of the same frequency travelling in opposite directions. Suitable conditions of wave interference may occur at the centre of a cyclonic depression or possibly if there is wave reflexion from a coast. In the latter case the microseisms are likely to be smaller, except perhaps locally. Confirmation of the present theory is provided by the observations of Bernard and Deacon, who discovered independently that the period of the microseisms is in many cases about half that of the ocean waves associated with them.

Vol. 243. A. 857. (Price 8*s*.) [Published 27 September 1950

1. Introduction

The word 'microseisms' is commonly used to denote the continuous oscillations of the ground of periods between 3 and 10 sec. which are recorded by all sensitive seismographs, and which are not due to earthquakes or to local causes such as rain, traffic or gusts of wind. Since the original researches of Bertelli in the latter half of the nineteenth century, many investigations have confirmed the close connexion of microseisms with disturbed weather conditions, especially with those centred over the sea. Increased microseismic activity tends to occur simultaneously over large areas of Europe or of North America (Gutenberg 1931, 1932; Lee 1934), and the greatest disturbance is found to be in a coastal region bordering on a well-developed depression. It is not true conversely (Whipple & Lee 1935) that depressions of the same intensity necessarily give rise to the same amplitude of microseisms. However, Ramirez (1940), by using a triangular arrangement of seismographs, has shown beyond doubt that microseisms at St Louis, Missouri, are received from the direction of depressions off the Atlantic coast. His methods of direction-finding have also formed the basis of a successful project for tracking hurricanes in the Caribbean area (Gilmore 1946).

Several suggestions as to the cause of microseisms have been put forward, none of which, however, is entirely satisfactory. Gherzi (1932) has considered microseisms to be due to 'pumping' of the atmosphere such as is sometimes shown on barographs near the centre of intense tropical cyclones. This cause cannot be excluded for storms of tropical intensity, where observations taken in the path of the storm show that the amplitude may be as much as 0·2 mm. of mercury (Bradford 1935). Ramirez, however, has pointed out (1940) that there is practically no connexion between the microseisms at St Louis and the barograph oscillations at St Louis or Florissant, even during the close passage of a tornado during March 1938. Also the periods of the oscillations quoted by Gherzi for the Shanghai typhoon are of several minutes, which would appear to be too long. It is considerably more doubtful whether microseisms could be caused by the much milder atmospheric oscillations found in temperate latitudes. The observations of Baird & Banwell in New Zealand (1940) have indicated amplitudes of only a few inches of air.

Scholte (1943) has sought to demonstrate that microseisms may be generated by atmospheric pressure on the surface of the sea, by showing that the amplitude of the compression waves generated by an oscillatory pressure spread sufficiently widely over the sea surface is as great as 10^{-4} times the amplitude of the gravity waves (ocean waves) so generated. The weakness of this argument is apparent. Ocean waves are not generated by oscillating pressure distributions of the type described by Scholte, but more probably by a systematic difference of pressure between the front and rear slopes of the crests of a wave train (Jeffreys 1925). The effect of a pressure distribution of this latter type, while tending continually to increase the energy of the gravity waves, would tend to cancel out for the much longer waves of compression.

An earlier theory, due originally to Wiechert and until recently strongly supported by Gutenberg, was that microseisms are caused by the impact of waves breaking against a steep coast. It is argued in favour of this theory that there is a statistical correlation between, for example, the amplitude of the microseisms at Hamburg and the height of the waves off the coast of Norway (Tams 1933). This theory will account for some of the facts, although it

involves a coefficient for the proportion of the wave energy imparted to the ground which some may consider too high (Bradford 1935). Observations also seem to show that microseisms associated with storms at sea may be recorded several hours before the waves reach the coast (Banerji 1930; Ramirez 1940; Deacon 1949), so that a further explanation, at any rate of these latter observations, is required.

Possibly the most natural explanation of microseisms, and one that might have been previously considered more seriously but for theoretical objections, is that they are generated by pressure variations on the sea bed due to ocean waves raised by the wind. It is unfortunate that in the past use has had to be made of Stokes's well-known theory of progressive waves, with the result that the pressure variations on the bottom, at any rate in deep water, appeared far too small (Gutenberg 1931; Whipple & Lee 1935). The physical reasons for this are twofold. In the first place the pressure variations in a progressive wave decrease exponentially with depth, and secondly the wave-length of gravity waves is extremely small compared to that of seismic waves, so that the contributions from different part of the sea bed effectively cancel one another. Banerji (1930) sought a way out by supposing that the water motion is not strictly irrotational, but his analysis cannot be defended. It was also shown (Whipple & Lee 1935) that the compressibility of the water makes little difference to the general result.† A further difficulty was that investigation of the wave periods usually showed them to be considerably greater than the corresponding periods of the microseisms. Bernard's careful studies of the periods of swell off the coast of Morocco (1937, 1941 *a*, *b*) indicated that they were in fact about twice the microseism periods. In a comparison of the Kew seismograms with records of waves taken at Perranporth in Cornwall, Deacon (1947) independently arrived at the same conclusion.

It has been pointed out (Longuet-Higgins & Ursell 1948) that Miche, in a theoretical study of wave motion (1944), discovered that the mean pressure on the bottom beneath a train of standing waves is not constant, as in a progressive wave, but fluctuates with an amplitude independent of the depth and proportional to the square of the wave height. This oscillation is of precisely the type required for the generation of ground movement, for not only is it unattenuated with depth (and is therefore the most important term at depths greater than about half a wave-length) but also, being in phase at all points of the bottom, it is suitable for producing long seismic waves. A further remarkable fact is that the frequency of this pressure variation is twice the fundamental frequency of the waves. Owing to the customary neglect of terms of higher order than the first, this term had been overlooked, the standing wave being in the first approximation the sum of two progressive waves of equal amplitudes travelling in opposite directions. A shorter proof of Miche's result, bringing to

† An attempt was made by Banerji (1935) to show that the compressibility of the water would allow pressure variations of the same period as the surface waves to be transmitted to depths great compared with the wave-length. However, an error in his analysis was pointed out by Whipple & Lee (1935, p. 295). In the same paper (1935) Banerji describes experiments in which he set up waves of length 2 to 6 cm. in tanks of depth 84 to 108 cm. and observed the oscillations in a tube of diameter 4 cm. sunk to varying depths and open at the lower end. Appreciable oscillations were observed at all depths. Banerji's results are difficult to interpret, but it seems unlikely that the compressibility of the water can have affected experiments on this scale. J. Darbyshire has also pointed out that the period of the oscillations shown in plates XXVII and XXVIII of Banerji's paper lies between 0·6 and 0·75 sec.; these cannot have been of the same period as the surface waves unless the latter were of length 56 to 88 cm., or comparable with the depth and width of the tank.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

light the physical reasons for the existence of this pressure oscillation, was given by Longuet-Higgins & Ursell (1948). A generalization of this proof led the present author to the conclusion (see § 3) that variations in the mean pressure over a wide area arise as a result of the interference of groups of waves of the same wave-length, but not necessarily of equal amplitude, travelling in opposite directions.

For a few years previously Bernard (1941 *a*, *b*) had held the view, unsupported at that time by hydrodynamical theory, that standing waves (Fr. *clapotis*) were the cause of microseisms. He had suggested that favourable conditions for standing waves would arise at the centre of a cyclonic depression or possibly off a steep coast where there was reflexion from the shore (this idea is to be distinguished from Wiechert's surf theory, although similar conditions would favour the generation of microseisms on either hypothesis). Bernard does not appear to have foreseen the doubling of the frequency of the unattenuated pressure variations in a standing wave, for he is inclined to suggest other causes for the difference between the frequencies of the microseisms and those of the waves (Bernard 1941 *a*, p. 10).

In the present paper we shall first investigate, in §§ 2 and 3, the physical reasons for the existence of unattenuated pressure variations of the type occurring in the standing wave and the general conditions under which they will occur; in § 4 the effect of the compressibility of the water on the wave motion will be considered; and in § 5, using the results of §§ 3 and 4, it will be shown that the second-order pressure variations due to surface waves are of the right order of magnitude for producing microseismic oscillations of the sea bed. We shall also consider briefly under what meteorological circumstances waves suitable for generating microseisms may be expected to be produced.

2. Pressure variations in a periodic wave train

2·1. *The attenuation of pressure variations and particle velocities with depth*

Although the second-order pressure variations in a standing wave in deep water are not attenuated exponentially with the depth, the unattenuated terms are not associated with any motion of the particles. That this is possible may be seen as follows. Let rectangular co-ordinates (x, y, z) be taken with the origin in the undisturbed level of the free surface and the z-axis vertically downwards. For simplicity we shall consider motion in two dimensions (x, z) only; similar arguments are, however, applicable to motion in three dimensions. We assume that the motion is irrotational, and that it is periodic in the x-direction with wave-length λ. The components of velocity (u, w) are given by

$$u = -\frac{\partial \phi}{\partial x}, \quad w = -\frac{\partial \phi}{\partial z}, \tag{1}$$

where, since the fluid is incompressible, we have

$$\nabla^2 \phi = -\left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}\right) = 0. \tag{2}$$

The equations of motion may be integrated (see Lamb 1932, § 20) to give the Bernoulli equation

$$\frac{p - p_s}{\rho} - gz = \frac{\partial \phi}{\partial t} - \tfrac{1}{2}(u^2 + w^2) + \theta(t), \tag{3}$$

where p denotes the pressure, ρ the density, g the acceleration of gravity, p_s the pressure at the free surface (supposed constant) and $\theta(t)$ is a function of the time t only. ϕ itself contains an arbitrary function of t; but this may be made definite by specifying that the mean value of ϕ with respect to x, taken over one wave-length, is zero. Similarly, by a suitable choice of axes the mean value of u may be made zero (both conditions may be satisfied for all values of z and t). Then, since ϕ is a harmonic function periodic in x and bounded when $z>0$, it may be shown that in water of infinite depth ϕ, u and w all diminish with z at least as rapidly as $\exp(-2\pi z/\lambda)$ (to all orders of approximation). Therefore when z exceeds about half a wave-length we have from equation (3)

$$\frac{p-p_s}{\rho}-gz=\theta(t). \tag{4}$$

Thus, although the particle velocities in any irrotational periodic motion must decrease exponentially with the depth, the pressure may still be a function of the time t. The pressure variation (4), being simultaneous over the whole fluid, is the same as if a uniform pressure $\theta(t)$ were applied to the free surface, the fluid being at rest. $\theta(t)$, being the limit of (3) when z tends to infinity, may be called the pressure variation at infinite depth. $\theta(t)$ does not in general vanish, though in one case, namely, that of the progressive wave, we may show that it is a constant; for in equation (3) every term except $\theta(t)$ is then a function of $(x-ct)$ and z, where c is the wave velocity. Therefore θ also is a function of $(x-ct)$. Hence θ, being independent of x, is independent of t also.

In general, since $\theta(t)$ is in phase at all points, there is a fluctuation in the mean pressure with respect to x on any plane $z=$ constant. Thus if $\bar{p}$ denote the mean pressure with respect to x in the interval $0\leqslant x\leqslant\lambda$ we have from (3)

$$\frac{\bar{p}-p_s}{\rho}-gz=-\frac{1}{\lambda}\int_0^\lambda \tfrac{1}{2}(u^2+w^2)\,dx+\theta(t) \tag{5}$$

(since the mean value of ϕ vanishes by hypothesis); and for large values of z we have

$$\frac{\bar{p}-p_s}{\rho}-gz=\theta(t). \tag{6}$$

The occurrence of an unattenuated pressure variation at infinite depth is therefore closely associated with a variation in the mean pressure on the plane $z=$ constant. As we saw in § 1, it is the variation in the mean pressure which is likely to be of physical importance in producing seismic oscillations of the sea bed.

2·2. *Evaluation of the mean pressure*

We shall now obtain a general expression for the mean pressure over a given area of the plane $z=$ constant, from which the special cases of the standing and the progressive wave may be very simply derived. It will not be assumed, in the first place, either that the motion is irrotational or periodic. Some of the equations will therefore be applicable to the more general types of motion to be discussed in § 3.

A very general relation between the vertical motion of a mass M of fluid consisting always of the same particles and the vertical forces acting upon it may be obtained as follows.

Suppose that (x, z) are rectangular co-ordinates referring always to the same particle of the fluid in the Lagrangian manner, so that x and z are functions of the time t and of the co-ordinates (x_0, z_0) at some fixed instant, say $t = 0$. The equation of motion in the vertical direction is

$$\frac{\partial p}{\partial z} - g\rho = -\frac{1}{\rho}\frac{\partial^2 z}{\partial t^2}, \tag{7}$$

and the equation of continuity may be expressed in the form

$$\rho\, dx\, dz = \rho_0\, dx_0\, dz_0, \tag{8}$$

where ρ_0 is the density when $t = 0$. Now we have

$$\int_M \rho \frac{\partial^2 z}{\partial t^2}\, dx\, dz = \int_M \rho_0 \frac{\partial^2 z}{\partial t^2}\, dx_0\, dz_0 = \frac{\partial^2}{\partial t^2}\int_M \rho_0 z\, dx_0\, dz_0 = \frac{\partial^2}{\partial t^2}\int_M \rho z\, dx\, dz. \tag{9}$$

Therefore on integrating equation (7) over the fluid M we find

$$\int_M \frac{\partial p}{\partial z}\, dx\, dz - \int_M g\rho\, dx\, dz = -\frac{\partial^2}{\partial t^2}\int_M \rho z\, dx\, dz. \tag{10}$$

In evaluating the integrals in equation (10) we may treat x, z and t as the independent variables, though the boundaries of M are now functions of t. The right-hand side of (10) may clearly be written $-\frac{1}{g}\frac{\partial^2 V}{\partial t^2}$, where V is the potential energy of the fluid M.

Suppose now that, in any wave motion at the free surface of an incompressible fluid, M denotes the body of fluid which at time $t = 0$ is contained between the free surface $z = \zeta$, the horizontal plane $z = z'$ and the two vertical planes $x = x_1$ and $x = x_2$. If p' denotes the pressure in the plane $z = z'$, and p_s the constant pressure at the free surface, we have, at the initial instant,

$$\int_M \frac{\partial p}{\partial z}\, dx\, dz = \int_{x_1}^{x_2} (p' - p_s)\, dx = (\bar{p}' - p_s)\,(x_2 - x_1), \tag{11}$$

where $\bar{p}'$ denotes the mean value of p' with respect to x. Similarly we have, since ρ is assumed to be constant,

$$\int_M g\rho\, dx\, dz = g\rho \int_{x_1}^{x_2} (z' - \zeta)\, dx = g\rho z'(x_2 - x_1) - g\rho \int_{x_1}^{x_2} \zeta\, dx. \tag{12}$$

To evaluate the third term in equation (10) we need an expression for the integral at times other than the initial instant. Suppose then that at time t the fluid M is bounded by the surfaces

$$z = \zeta(x, t), \quad z = z' + \zeta'(x, t), \quad x = \xi_1(z, t) \quad \text{and} \quad x = \xi_2(z, t),$$

where

$$\zeta'(x, 0) = 0, \quad \xi_1(z, 0) = x_1, \quad \xi_2(z, 0) = x_2. \tag{13}$$

The (x, z) co-ordinates of the intersections of the surfaces $z = \zeta'$, $(z' + \zeta')$ with the surfaces $x = \xi_1, \xi_2$ may be denoted by (α_1, γ_1), (α_1', γ_1'); (α_2, γ_2), (α_2', γ_2') respectively, these being functions of t. Then we have

$$\int_M z\, dx\, dz = \int_{\alpha_1'}^{\alpha_2'} \tfrac{1}{2}(z' + \zeta')^2\, dx - \int_{\alpha_1}^{\alpha_2} \tfrac{1}{2}\zeta^2\, dx + \int_{\gamma_2}^{\gamma_2'} \xi_2 z\, dz - \int_{\gamma_1}^{\gamma_1'} \xi_1 z\, dz - \tfrac{1}{2}[\alpha_2'\gamma_2'^2 - \alpha_1'\gamma_1'^2 - \alpha_2\gamma_2^2 + \alpha_1\gamma_1^2]. \tag{14}$$

On differentiating twice with respect to t we find

$$\frac{\partial^2}{\partial t^2}\int_M z\, dx\, dz = \int_{\alpha_1'}^{\alpha_2'} \frac{\partial^2}{\partial t^2}\tfrac{1}{2}(z' + \zeta')^2\, dx - \int_{\alpha_1}^{\alpha_2} \frac{\partial^2}{\partial t^2}\tfrac{1}{2}\zeta^2\, dx + \int_{\gamma_2}^{\gamma_2'} \ddot{\xi}_2 z\, dz - \int_{\gamma_1}^{\gamma_1'} \ddot{\xi}_1 z\, dz$$
$$+ 2[\dot{\alpha}_2'\dot{\gamma}_2'\gamma_2' - \dot{\alpha}_1'\dot{\gamma}_1'\gamma_1' - \dot{\alpha}_2\dot{\gamma}_2\gamma_2 + \dot{\alpha}_1\dot{\gamma}_1\gamma_1], \tag{15}$$

where a dot denotes partial differentiation with respect to t. At the initial instant we have

$$\alpha_1 = \alpha_1' = x_1, \quad \alpha_2 = \alpha_2' = x_2, \quad \gamma_1' = \gamma_2' = z'. \tag{16}$$

Therefore, if ζ_1 and ζ_2 denote the values of ζ when $x = x_1$ and x_2, equation (10) becomes

$$\frac{\bar{p}' - p_s}{\rho} - gz' = \frac{1}{x_2 - x_1}\int_{x_1}^{x_2}\left[\frac{\partial^2}{\partial t^2}(\tfrac{1}{2}\zeta^2 - \tfrac{1}{2}\zeta'^2) - z'\ddot{\xi} - g\zeta\right]dx$$
$$- \frac{1}{x_2 - x_1}\left[\int_{\zeta_2}^{z'} \ddot{\xi}_2 z\,dz - \int_{\zeta_1}^{z'} \ddot{\xi}_1 z\,dz\right] - \frac{2}{x_2 - x_1}[\dot{\alpha}_2'\dot{\gamma}_2'\gamma_2' - \dot{\alpha}_1'\dot{\gamma}_1'\gamma_1' - \dot{\alpha}_2\dot{\gamma}_2\gamma_2 + \dot{\alpha}_1\dot{\gamma}_1\gamma_1]. \tag{17}$$

The above equation may be put into a form which is independent of the initial instant chosen. For if (u', w') denote the components of velocity in the plane $z = z'$ we have, at the initial instant,

$$\frac{\partial^2}{\partial t^2}(\tfrac{1}{2}\zeta'^2) = \zeta'\ddot{\zeta}' + \dot{\zeta}'^2 = w'^2. \tag{18}$$

Also by considering $D^2(\zeta' - z)/Dt^2$, where D/Dt denotes differentiation following the motion, we find

$$\ddot{\zeta}' = \dot{w}' - \frac{\partial}{\partial x}(u'w'). \tag{19}$$

Similarly
$$\ddot{\xi}_i = \dot{u}_i - \frac{\partial}{\partial z}(u_i w_i) \quad (i = 1, 2), \tag{20}$$

where (u_i, w_i) are the velocity components in the plane $x = x_i$. Since $(\dot{\alpha}_i, \dot{\gamma}_i)$ and $(\dot{\alpha}_i', \dot{\gamma}_i')$ are equal to the components of velocity at (x_i, ζ) and (x_i, z'), we have finally, after integrating by parts and dropping the dashes,

$$\frac{\bar{p} - p_s}{\rho} - gz = \frac{1}{x_2 - x_1}\int_{x_1}^{x_2}\left[\frac{\partial^2}{\partial t^2}(\tfrac{1}{2}\zeta^2) - w^2 - z\dot{w} - g\zeta\right]dx - \frac{1}{x_2 - x_1}\left[\int_{\zeta}^{z}(\dot{u}z + uw)\,dz - (uwz)_{z=\zeta}\right]_{x=x_1}^{x=x_2}. \tag{21}$$

The above equation is now valid for all values of z and t. In equation (21) the first group of terms would give the mean pressure on the plane $z =$ constant if the planes $x = x_1, x_2$ were assumed to be vertical barriers. The second group of terms gives the correction due to the motion across these planes.

By allowing x_2 to tend to x_1 in equation (21) an expression for the pressure at any given point can be obtained. Thus

$$\frac{p - p_s}{\rho} - gz = \frac{\partial^2}{\partial t^2}(\tfrac{1}{2}\zeta^2) - w^2 - z\dot{w} - g\zeta - \frac{\partial}{\partial x}\left[\int_{\zeta}^{z}(\dot{u}z + uw)\,dz - (uwz)_{z=\zeta}\right]. \tag{22}$$

It may easily be verified that in a periodic wave motion in deep water the first-order terms on the right-hand side of (22) decrease exponentially with the depth.

Suppose now that the motion is periodic in x with wave-length λ. If we write $x_1 = 0$, $x_2 = \lambda$ in equation (21) the second group of terms then vanishes identically. Further, if the origin is assumed to be in the mean surface level we have

$$\int_0^{\lambda} g\zeta\,dx = 0; \tag{23}$$

and since the net flow of water across the plane $z =$ constant is zero we have also

$$\int_0^\lambda z\dot{w}\,dx = z\frac{\partial}{\partial t}\int_0^\lambda w\,dx = 0. \tag{24}$$

Therefore the mean pressure over one wave-length is given by

$$\frac{\bar{p}-p_s}{\rho} - gz = \frac{1}{\lambda}\frac{\partial^2}{\partial t^2}\int_0^\lambda \tfrac{1}{2}\zeta^2\,dx - \frac{1}{\lambda}\int_0^\lambda w^2\,dx. \tag{25}$$

If, in addition, the motion is irrotational we find by comparison with (5) that the function $\theta(t)$ is given by

$$\theta(t) = \frac{1}{\lambda}\frac{\partial^2}{\partial t^2}\int_0^\lambda \tfrac{1}{2}\zeta^2\,dx + \frac{1}{\lambda}\int_0^\lambda \tfrac{1}{2}(u^2-w^2)\,dx. \tag{26}$$

It may be verified that the second term is independent of z, for

$$\frac{\partial}{\partial z}\int_0^\lambda (u^2-w^2)\,dx = \int_0^\lambda \left(u\frac{\partial w}{\partial x} + w\frac{\partial u}{\partial x}\right)dx = \Big[uw\Big]_0^\lambda, \tag{27}$$

which vanishes by the periodicity of the motion. In deep water, since u and w decrease exponentially with depth, the pressure variation at infinite depth is given by

$$\theta(t) = \frac{1}{\lambda}\frac{\partial^2}{\partial t^2}\int_0^\lambda \tfrac{1}{2}\zeta^2\,dx. \tag{28}$$

In water of constant finite depth h the vertical velocity w vanishes when $z = h$. From (25) we see that the mean pressure variation on the bottom is also given by the right-hand side of (28). Thus both the pressure variation at infinite depth and the mean pressure on the bottom in the case of constant finite depth, depend on a second-order function of the wave amplitude, closely associated with changes in the potential energy of the wave train.

It will be noticed that the equations so far obtained are exact, and that no assumptions depending on the smallness of the wave amplitude have been made.

2·3. *The standing wave and progressive wave*

We shall now use the formulae of the previous section to evaluate the mean pressure on the bottom in some special cases of wave motion. This may be done, as we shall see, by consideration of the first approximation only.

Let the water be of constant depth h. Consider a motion which in the first approximation consists of two progressive waves of equal wave-length λ and period T travelling in opposite directions. The equation of the free surface is given by

$$\zeta = a_1\cos(kx-\sigma t) + a_2\cos(kx+\sigma t) + O(a^2k), \tag{29}$$

where $k = 2\pi/\lambda$, $\sigma = 2\pi/T$ and

$$\sigma^2 = gk\tanh kh \tag{30}$$

(Lamb 1932, p. 364). The last term in equation (29) represents a remainder of second or higher order in the wave amplitudes a_1 and a_2 which it will not be necessary to evaluate. When $z = h$, w vanishes, and so from equation (25) the mean pressure $\bar{p}_h$ on the bottom is given by

$$\begin{aligned}\frac{\bar{p}_h-p_s}{\rho} - gh &= \frac{1}{\lambda}\frac{\partial^2}{\partial t^2}\int_0^\lambda \tfrac{1}{2}[a_1\cos(kx-\sigma t) + a_2\cos(kx+\sigma t)]^2\,dx + O(a^3\sigma^2k^2)\\ &= \frac{\partial^2}{\partial t^2}\tfrac{1}{4}(a_1^2+a_2^2+2a_1a_2\cos 2\sigma t) + O(a^3\sigma^2k^2)\\ &= -2a_1a_2\sigma^2\cos 2\sigma t,\end{aligned} \tag{31}$$

to the second order of approximation. Thus the mean pressure fluctuation on the bottom is of twice the frequency of the waves and proportional to the product of the wave amplitudes. For a given period T it is also independent of the depth h.

Two special cases are of interest. First, when the amplitude of one of the opposing waves is zero, that is, in the case of a single progressive wave, the right-hand side of equation (31) vanishes. The mean pressure on the bottom is therefore constant. Secondly, when the amplitudes of the two waves are equal and

$$a_1 = a_2 = \tfrac{1}{2}a, \tag{32}$$

say, we have a standing wave given by

$$\zeta = a \cos kx \cos \sigma t + O(a^2 k). \tag{33}$$

From equation (31) we have then

$$\frac{\bar{p}_h - p_s}{\rho} - gh = -\tfrac{1}{2}a^2\sigma^2 \cos 2\sigma t. \tag{34}$$

Therefore in a standing wave the mean pressure on the bottom varies with twice the frequency of the original wave and with an amplitude proportional to the square of the wave amplitude.

Equation (34) was obtained by Miche (1944, p. 73, equation (85)) after evaluating the second approximation to the wave motion in full.

A physical explanation of these two results, and of the difference between them, may be given as follows. Consider first the standing wave given by equation (33). When $t = (n+\frac{1}{2})\,T$, n being an integer, the wave surface is approximately flat. The centre of gravity of the whole wave train is therefore at its lowest point. On the other hand, when $t = nT$ the wave crests are fully formed and the centre of gravity has risen, since water has been transferred from below to above the mean level (this is equivalent to saying that the potential energy is increased). This raising and lowering of the centre of gravity occurs twice in a complete cycle. But the vertical motion of the centre of gravity of any mass of fluid is determined solely by the vertical external forces acting upon it. Of these, the force due to gravity is constant, and the pressure on the free surface supplies a constant additional downwards force. There remains the pressure on the bottom, which must therefore fluctuate in a similar manner, with twice the frequency of the waves.

In a progressive wave, on the other hand, similar considerations show that the mean pressure on the bottom is constant. For the potential energy, and hence also the centre of mass, of the whole wave train remains at a constant level throughout. There can be therefore no fluctuation in the mean pressure on the bottom.

It should be possible to verify formulae (31) and (34) quite simply by experiment, since these terms represent the only pressure variations measurable at a depth of more than half a wave-length. The water should be almost still at this depth, so that the formation of eddies round the measuring apparatus would be avoided. A standing wave could be produced in a long wave tank by the reflexion of a wave train from a vertical barrier at one end of the tank. If the inclination of the barrier to the horizontal were varied, reflected waves of different amplitude would be obtained, since for small inclinations some energy would almost certainly be absorbed at the barrier itself. In the first-order theory of surface waves the absorption of energy at the barrier cannot be taken into account without assuming a

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

singularity at the origin, and the amount of energy absorbed is indeterminate. However, by the present method the coefficient of reflexion could be determined experimentally, since the pressure variation on the bottom (at a few wave-lengths from the barrier) is directly proportional to the amplitude of the reflected wave. Hence also some indication could probably be obtained as to the amount of wave reflexion taking place at a steep coast and from beaches of different gradients.

3. General types of wave motion

Perfectly periodic wave trains of standing or progressive type rarely occur in practice, and in the present section we shall consider the pressure variation in wave motions of more general type. When the motion is not perfectly periodic in space the pressure variation at infinite depth, in the sense of § 2·1, no longer exists, but expressions may still be found for the mean pressure or the total force over a given area of the plane $z =$ constant. These assume a simple form provided that the area is large enough for the motion across the boundaries to become negligible.

3·1. *The force on a given area of the plane $z =$ constant*

Still considering motion in two dimensions only, let $\mathbf{F}$ denote the variable part of the total force, per unit distance in the y-direction, acting on the plane $z = z'$ in the interval $-R<x<R$, i.e.

$$\frac{\mathbf{F}}{\rho} = 2R\left(\frac{\bar{p}-p_s}{\rho}-gz\right), \tag{35}$$

where $\bar{p}$ is the mean pressure on the plane $z = z'$ in this interval. Then from equation (21) we have

$$\frac{\mathbf{F}}{\rho} = \int_{-R}^{R}\left[\frac{\partial^2}{\partial t^2}(\tfrac{1}{2}\zeta^2)-w^2-z\dot{w}-g\zeta\right]dx-\left[\int_{\zeta}^{z}(\dot{u}z+uw)\,dz-(uwz)_{z=\zeta}\right]_{-R}^{R}. \tag{36}$$

Now since the flow of water across the horizontal plane $z = z'$ $(-R<x<R)$ is equal to the net flow across the vertical planes $x = \pm R$ $(z>z')$, we have

$$\int_{-R}^{R} z\dot{w}\,dx = \left[z\int_{z}^{h}\dot{u}\,dz\right]_{-R}^{R}, \tag{37}$$

where h denotes the depth of water (not necessarily constant); if the depth is supposed infinite, the upper limit of the integral must be replaced by ∞. Similarly, if the mean level of the free surface $z = \zeta$ is zero at time $t = 0$ we have

$$\int_{-R}^{R} g\zeta\,dx = \left[\int_{0}^{t}dt\int_{\zeta}^{h}gu\,dz\right]_{-R}^{R}. \tag{38}$$

Hence from equation (36), after integrating by parts,

$$\frac{\mathbf{F}}{\rho} = \int_{-R}^{R}\left[\frac{\partial^2}{\partial t^2}(\tfrac{1}{2}\zeta^2)-w^2\right]dx-\left[z\int_{\zeta}^{h}\dot{u}\,dz-g\int_{0}^{t}dt\int_{\zeta}^{h}u\,dz\right]_{-R}^{R}$$
$$-\left[\int_{\zeta}^{h}\left\{\int_{\zeta}^{z}\dot{u}\,dz+uw\right\}dz-(uwz)_{z=\zeta}\right]_{-R}^{R}. \tag{39}$$

Let us consider the relative magnitudes of the terms in equation (39). We suppose that the motion is wave-like, in the sense that the energy is nearly all confined to a narrow range of frequencies in the frequency spectrum (as defined in § 3·2); and that the mean frequency $\sigma/2\pi$ corresponds to a wave-length λ which is small compared with R. In general, the relative phase of the motion at two widely separated points of the x-axis will be random. We may, however, suppose that the motion is regular and periodic over any interval of the x-axis less than or equal to $2R_1$, say. We suppose also that the motion is initially confined to an interval $-R_2<x<R_2$ (where R_2 may be very great compared with R_1), that is, that the elevation and vertical velocity of the free surface at points outside this interval are initially zero. There will be three distinct cases:

Case 1. $R\leqslant R_1$, i.e. the motion is regular over the whole interval $-R<x<R$. Then

$$\int_{-R}^{R}\left[\frac{\partial^2}{\partial t^2}(\tfrac{1}{2}\zeta^2)-w^2\right]dx \tag{40}$$

is of order $a^2\sigma^2R$, where a is the maximum wave elevation. If we assume for the moment that u and w are of order $a\sigma$ and that

$$\left[\int_{\zeta}^{z}u\,dz\right]_{-R}^{R} \tag{41}$$

is of order $a\sigma\lambda$ for all z, the remaining terms in (39) are of order $a\sigma^2\lambda z$ or $a\sigma^2\lambda^2$ at the most (if g is of order $\lambda\sigma^2$). Hence if R/k and R/z were sufficiently large we should have

$$\frac{\mathbf{F}}{\rho}=\int_{-R}^{R}\left[\frac{\partial^2}{\partial t^2}(\tfrac{1}{2}\zeta^2)-w^2\right]dx \tag{42}$$

approximately. It must, however, be verified that these second-order pressure variations, which are in phase over the whole interval, do not produce any significant motion across the planes $x=\pm R$. Now if we consider the displacement produced by the pressure distribution

$$\frac{p}{\rho}=\begin{cases}2a^2\sigma^2\cos 2\sigma t & (|x|<R),\\ 0 & (|x|>R),\end{cases} \tag{43}$$

acting on the upper surface of deep water we find that the velocities in the planes $x=\pm R$ are of order $a^2\sigma/\lambda$ (we ignore a logarithmic singularity at $z=0$, which is due to the local discontinuity in pressure), and that the total flow (41) is of order $a^2\sigma\log(R/\lambda)$. The assumption that (41) is of order independent of R therefore needs slight modification in this case, but since $\log(R/\lambda)$ is small compared with R/λ the validity of equation (42) is not affected.

When z is small compared with λ the approximation (42) is valid under the condition $Ra/\lambda^2\ll 1$. However, the first-order terms in (39), taken together, may be expected to decrease rapidly with the depth, and when z is greater than about $\frac{1}{2}\lambda$ the largest terms in the remainder will arise from the unattenuated pressure variations of second order. Hence (42) will be valid under the less restrictive conditions $R/\lambda\gg 1$ and $R/z\gg 1$. Since the second term in (42) will be small compared with the first we shall then have

$$\frac{\mathbf{F}}{\rho}=\frac{\partial^2}{\partial t^2}\int_{-R}^{R}\tfrac{1}{2}\zeta^2\,dx. \tag{44}$$

In particular (44) will be valid if z is of order λ and $R/\lambda\gg 1$.

Case 2. $R_1<R\leqslant R_2$. In this case suppose the interval $-R<x<R$ to be divided into smaller intervals of length less than or equal to $2R_1$. We assume that the motion in each of the smaller

intervals is regular but that the phase differences between successive intervals are random. Since the sum of n vectors of comparable magnitude in random-phase relationship with one another increases like $n^{\frac{1}{2}}$ the integral (40) will be of order $a^2\sigma^2 R_1(R/R_1)^{\frac{1}{2}}$. If we assume that the velocities are bounded and that the total flow across any plane $x =$ constant is of order $a\sigma\lambda$ or $a^2\sigma \log(R_1/\lambda)$ at most, equations (42) and (44) will be valid under conditions similar to case 1; in particular, (44) will hold if z is of order λ and $(RR_1)^{\frac{1}{2}}/\lambda \gg 1$.

Case 3. $R > R_2$. By allowing R to tend to infinity an exact expression for the total force $\mathbf{F}$ over the whole plane $z =$ constant may be obtained. The velocity potential of the motion due to an initial elevation of the free surface concentrated in the line $x = z = 0$ is proportional to $gtz(x^2+z^2)^{-1}$, when $gt^2(x^2+z^2)^{-\frac{1}{2}}$ is small (see Lamb 1932, § 238). A similar result will hold when the initial disturbance is distributed over a finite interval of the x-axis. Hence for very large R the velocities across the planes $x = \pm R$ will initially be proportional to R^{-2}, and the total flow (41) will be proportional to R^{-1}. The terms in (39) to be evaluated at the planes $x = \pm R$ therefore tend to zero. But since the total potential energy is finite, we may assume that the first integral in (39) converges. Hence the total force $\mathbf{F}$ over the whole plane is given by

$$\frac{\mathbf{F}}{\rho} = \int_{-\infty}^{\infty} \left[\frac{\partial^2}{\partial t^2}(\tfrac{1}{2}\zeta^2) - w^2\right] dx. \tag{45}$$

When z is greater than about $\frac{1}{2}\lambda$ the second term in the integrand will be small compared with the first, so that

$$\frac{\mathbf{F}}{\rho} = \frac{\partial^2}{\partial t^2}\int_{-\infty}^{\infty} \tfrac{1}{2}\zeta^2 dx \tag{46}$$

approximately.

The previous results may be extended without difficulty to motion in three dimensions. Let $\bar{p}$ denote the mean pressure on the plane $z =$ constant inside the square S given by $-R < x < R$, $-R < y < R$, and let $\mathbf{F}$ denote the variable part of the total force acting on the plane inside S, i.e.

$$\frac{\mathbf{F}}{\rho} = 4R^2\left(\frac{\bar{p} - p_s}{\rho} - gz\right). \tag{47}$$

If the motion inside S is assumed to be wave-like with mean wave-length λ then we may establish that

$$\frac{\mathbf{F}}{\rho} = \int_{-R}^{R}\int_{-R}^{R} \left[\frac{\partial^2}{\partial t^2}(\tfrac{1}{2}\zeta^2) - w^2\right] dx\, dy \tag{48}$$

under similar conditions; in particular, if z is comparable with λ, and R/λ and $(RR_1)^{\frac{1}{2}}/\lambda$ are both large compared with unity, where $2R_1$ in the side of the largest square over which the second-order pressure variations are effectively in phase. Since the motion diminishes rapidly with depth, we shall have in this case also

$$\frac{\mathbf{F}}{\rho} = \frac{\partial^2}{\partial t^2}\int_{-R}^{R}\int_{-R}^{R} \tfrac{1}{2}\zeta^2 dx\, dy. \tag{49}$$

If it is supposed that the motion is initially confined to a finite region of the (x, y) plane we may show that the motion produces a force $\mathbf{F}$ over the whole plane given by

$$\frac{\mathbf{F}}{\rho} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left[\frac{\partial^2}{\partial t^2}(\tfrac{1}{2}\zeta^2) - w^2\right] dx\, dy. \tag{50}$$

Again, when z is greater than about $\frac{1}{2}\lambda$ we have approximately

$$\frac{\mathbf{F}}{\rho} = \frac{\partial^2}{\partial t^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tfrac{1}{2}\zeta^2 dx\, dy. \tag{51}$$

3·2. *The two-dimensional frequency spectrum*

In order to be able to describe the motion of the sea surface in terms of its frequency characteristics, we shall now introduce the two-dimensional frequency spectrum. The mean pressure, or total force, over a large area may be derived immediately from the frequency spectrum owing to the connexion of the mean pressure with the potential energy of the waves.

Any continuous and absolutely integrable function $f(x, y)$ of two variables may be expressed in the form

$$f(x,y) = \Re\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v)\, e^{i(ukx+vky)}\, du\, dv \tag{52}$$

or

$$f(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tfrac{1}{2}[F(u,v)+F^*(-u,-v)]\, e^{i(ukx+vky)}\, du\, dv, \tag{53}$$

where

$$\tfrac{1}{2}[F(u,v)+F^*(-u,-v)] = (k/2\pi)^2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, e^{-i(ukx+vky)}\, dx\, dy, \tag{54}$$

provided that the right-hand side of (54) is also absolutely integrable (Bochner 1932, § 44). In the above equations $\Re$ denotes the real part and F^* denotes the conjugate complex function of F. The value of

$$\tfrac{1}{2}[F(u,v)-F^*(-u,-v)] \tag{55}$$

is still indeterminate.

Let $z = \zeta$ be the equation of the free surface in any wave motion in two horizontal dimensions. We shall assume the general conditions necessary for the validity of the following work, and in particular the possibility of differentiating under the integral sign. Suppose then that the values of ζ and $\partial\zeta/\partial t$ at the initial instant $t = 0$ are expanded in the forms

$$(\zeta)_{t=0} = \Re\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A\, e^{i(ukx+vky)}\, du\, dv, \tag{56}$$

$$\left(\frac{\partial\zeta}{\partial t}\right)_{t=0} = \Re\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} B\, e^{i(ukx+vky)}\, du\, dv, \tag{57}$$

A and B being functions of (u, v). We may impose the further condition

$$B = i\sigma A \tag{58}$$

where σ is the positive function of u and v given by

$$\sigma^2 = (u^2+v^2)^{\frac{1}{2}}\, gk \tanh (u^2+v^2)^{\frac{1}{2}}\, kh. \tag{59}$$

By equation (54) we have then, using equation (58),

$$\tfrac{1}{2}(A+A_-^*) = (k/2\pi)^2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (\zeta)_{t=0}\, e^{-i(ukx+vky)}\, dx\, dy, \tag{60}$$

$$\tfrac{1}{2}(i\sigma A - i\sigma A_-^*) = (k/2\pi)^2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(\frac{\partial\zeta}{\partial t}\right)_{t=0} e^{-i(ukx+vky)}\, dx\, dy, \tag{61}$$

where A_- denotes $A(-u, -v)$. These last equations are equivalent to the single equation

$$A = (k/2\pi)^2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(\zeta + \frac{1}{i\sigma}\frac{\partial\zeta}{\partial t}\right)_{t=0} e^{-i(ukx+vky)}\, dx\, dy. \tag{62}$$

Consider now the expression

$$\eta = \Re\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A\, e^{i(ukx+vky+\sigma t)}\, du\, dv, \tag{63}$$

where A is determined by (62). The expression under the integral sign represents a wave whose crests are parallel to the line

$$ux+vy=0, \tag{64}$$

and whose wave-length λ is given by

$$\lambda=\frac{2\pi}{(u^2+v^2)^{\frac{1}{2}}k}. \tag{65}$$

By equation (59) this wave satisfies the period equation for waves in water of constant depth h, and hence η is also a solution, to the first order of approximation. But from (5) and (57) we have

$$(\zeta)_{t=0}=(\eta)_{t=0}, \quad \left(\frac{\partial\zeta}{\partial t}\right)_{t=0}=\left(\frac{\partial\eta}{\partial t}\right)_{t=0}. \tag{66}$$

Now an irrotational motion is uniquely determined by the initial values of the surface elevation and its rate of change with time (for the difference between two motions with the same initial conditions has initially no kinetic or potential energy). It follows that $\zeta=\eta$, i.e.

$$\zeta=\Re\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}A\,\mathrm{e}^{i(ukx+vky+\sigma t)}\,du\,dv \tag{67}$$

for all times t.

Any given free motion of the sea surface may therefore be analyzed (in the first approximation) into the sum of a number of wave components of all possible wave-lengths and travelling in all possible directions. This analysis, by equation (62), is unique. Each wave component corresponds to a vector $\overline{OP}$ in the (x, y) plane drawn from the origin to the point $P(-uk, -vk)$. The direction of $\overline{OP}$ gives the direction of propagation of the wave, and the length of $\overline{OP}$ is, from equation (65), equal to 2π divided by the wave-length. Wave components of the same length will correspond to points P lying on the same circle centre O, and diametrically opposite points will correspond to wave components of the same wave-length but travelling in opposite directions. Such pairs of wave components play an important part in the following theory and will be called opposite wave components.

Equation (67) may also be written in the form

$$\zeta=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\tfrac{1}{2}(A\,\mathrm{e}^{i\sigma t}+A_-^*\,\mathrm{e}^{-i\sigma t})\,\mathrm{e}^{i(ukx+vky)}\,du\,dv. \tag{68}$$

Hence by an extension of the Parseval-Plancherel theorem (Bochner 1932, §§ 41·5 and 44·8) we have

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\zeta^2\,dx\,dy=(2\pi/k)^2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\tfrac{1}{2}(A\,\mathrm{e}^{i\sigma t}+A_-^*\,\mathrm{e}^{-i\sigma t})|^2\,du\,dv, \tag{69}$$

since the integral on the left-hand side is convergent. After simplifying the right-hand side, we have

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\tfrac{1}{2}\zeta^2\,dx\,dy=\Re(\pi/k)^2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(AA^*+AA_-\,\mathrm{e}^{2i\sigma t})\,du\,dv. \tag{70}$$

Thus the potential energy of the motion is given by

$$\Re\rho g(\pi/k)^2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(AA^*+AA_-\,\mathrm{e}^{2i\sigma t})\,du\,dv. \tag{71}$$

Similarly, we find for the kinetic energy

$$\Re\rho g(\pi/k)^2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(AA^*-AA_-\,\mathrm{e}^{2i\sigma t})\,du\,dv, \tag{72}$$

and so the total energy is given by

$$2\rho g(\pi/k)^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} AA^* \, du\, dv \tag{73}$$

(the above integral being real). The total energy therefore depends only upon the square of the modulus of the wave amplitude $A(u, v)$. On the other hand, both the potential and the kinetic energies separately vary with the time and depend on the product AA_-.

3·3. *Pressure variations in terms of the frequency spectrum*

We are now in a position to determine the general conditions for a variation in the mean pressure or total force acting on a large area of the plane $z =$ constant. We consider first the simpler case when the area includes the whole (x, y) plane.

From equations (68) and (69) we have

$$\begin{aligned}\frac{\mathbf{F}}{\rho} &= \Re(\pi/k)^2 \frac{\partial^2}{\partial t^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (AA^* + AA_- \, \mathrm{e}^{2i\sigma t})\, du\, dv \\ &= -\Re 4(\pi/k)^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} AA_- \sigma^2 \, \mathrm{e}^{2i\sigma t}.\end{aligned} \tag{74}$$

Now A and A_- are the complex amplitudes of opposite wave-conponents in the frequency-spectrum. It follows from (74) that

(1) Variations in $\mathbf{F}$ arise only from opposite pairs of wave components in the frequency spectrum.

(2) The contribution to $\mathbf{F}$ from any opposite pair of wave components is of twice their frequency and proportional to the product of their amplitudes.

(3) The total force $\mathbf{F}$ is the integrated sum of the contributions from all opposite pairs of wave components separately.

A wave group may be defined as a motion in which most of the energy is confined to a small region of the (u, v) plane, excluding the origin. Thus a single group of waves will not cause variations in the total force $\mathbf{F}$. In order that $\mathbf{F}$ should be appreciable the motion must contain at least two wave groups which are opposite, in the sense that some wave components of the first group are opposite to some wave components of the second.

In practice we must consider the force $\mathbf{F}$ over only a finite region of the (x, y) plane. Let this be the square S $(-R<x<R,\ -R<y<R)$. We define a hypothetical motion ζ' such that at any time ζ' and $\partial\zeta'/\partial t$ are equal to the corresponding values of ζ and $\partial\zeta/\partial t$ inside S and zero outside. This motion will not satisfy the equations of motion, especially near the boundaries of S, but we shall now have

$$\frac{\mathbf{F}}{\rho} = \frac{\partial^2}{\partial t^2}\int_{-R}^{R}\int_{-R}^{R} \tfrac{1}{2}\zeta^2 \, dx\, dy = \frac{\partial^2}{\partial t^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tfrac{1}{2}\zeta'^2 \, dx\, dy. \tag{75}$$

We also define $A'(u, v; t)$ by the equations

$$\left.\begin{aligned}\zeta' &= \Re\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A' \, \mathrm{e}^{i(ukx+vky+\sigma t)}\, du\, dv, \\ \frac{\partial\zeta'}{\partial t} &= \Re\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} i\sigma A' \, \mathrm{e}^{i(ukx+vky+\sigma t)}\, du\, dv.\end{aligned}\right\} \tag{76}$$

Then we have, as before,

$$A' \, \mathrm{e}^{i\sigma t} = (k/2\pi)^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(\zeta' + \frac{1}{i\sigma}\frac{\partial\zeta'}{\partial t}\right) \mathrm{e}^{-i(ukx+vky)}\, dx\, dy. \tag{77}$$

If the actual motion is given by equation (67) we have on substitution in (77)

$$A'(u,v;t) = (k/2\pi)^2 \int_{-R}^{R}\int_{-R}^{R} dx\,dy \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} du_1\,dv_1$$
$$\times \frac{1}{2}\Big[A(u_1,v_1)\left(1+\frac{\sigma_1}{\sigma}\right)\mathrm{e}^{-i[(u-u_1)kx+(v-v_1)ky+(\sigma-\sigma_1)t]}$$
$$+A^*(u_1,v_1)\left(1-\frac{\sigma_1}{\sigma}\right)\mathrm{e}^{-i[(u+u_1)kx+(v+v_1)ky+(\sigma+\sigma_1)t]}\Big], \tag{78}$$

where σ is written for $\sigma(u_1,v_1)$. Since k is still at our disposal we may put

$$2\pi/k = 2R. \tag{79}$$

Then, after integration with respect to x and y, we find

$$A'(u,v;t) = \frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A(u_1,v_1)\left(1+\frac{\sigma_1}{\sigma}\right)\frac{\sin(u-u_1)\pi}{(u-u_1)\pi}\frac{\sin(v-v_1)\pi}{(v-v_1)\pi}\mathrm{e}^{-i(\sigma-\sigma_1)t}\,du_1\,dv_1$$
$$+\frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A^*(u_1,v_1)\left(1-\frac{\sigma_1}{\sigma}\right)\frac{\sin(u+u_1)\pi}{(u+u_1)\pi}\frac{\sin(v+v_1)\pi}{(v+v_1)\pi}\mathrm{e}^{-i(\sigma+\sigma_1)t}\,du_1\,dv_1$$
$$= I_1+I_2, \tag{80}$$

say. Now by hypothesis the frequency spectrum of ζ consists chiefly of waves whose wave-length, given by (65), is small compared with $2R$. From (79) it follows that $A(u_1,v_1)$ is appreciably large only when $(u_1^2+v_1^2)^{\frac{1}{2}}$ is large. But the factors in the denominators of I_1 and I_2 make the integrands small except when $(u_1,v_1) \doteqdot (u,v)$ in the first case and $(u_1,v_1) \doteqdot (-u_1,-v_1)$ in the second. In either case $\sigma_1 \doteqdot \sigma$, so that the contribution from I_2 is small, while that from I_1 gives

$$A'(u,v;t) \doteqdot \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A(u_1,v_1)\frac{\sin(u-u_1)\pi}{(u-u_1)\pi}\frac{\sin(v-v_1)\pi}{(v-v_1)\pi}\mathrm{e}^{-i(\sigma-\sigma_1)t}\,du_1\,dv_1. \tag{81}$$

Although A' is dependent upon t, the integrals for $\partial A'/\partial t$, $\partial^2 A'/\partial t^2$, ... contain factors $(\sigma-\sigma_1)$, $(\sigma-\sigma_1)^2$, ... which are small over the critical range of integration near (u,v). These expressions are therefore small, and A' is only a slowly varying quantity.

From equations (75) we have then

$$\frac{\mathbf{F}}{\rho} = \Re(\pi/k)^2\frac{\partial^2}{\partial t^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(A'A'^*+A'A'_-\,\mathrm{e}^{2i\sigma t})\,du\,dv$$
$$= -\Re 4(\pi/k)^2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A'A'_-\,\sigma^2\,\mathrm{e}^{2i\sigma t}\,du\,dv. \tag{82}$$

The expression for the force $\mathbf{F}$ over a finite area is therefore similar to that over the whole plane, except that the original spectrum A is replaced by the new spectrum A'. Equation (81) shows that A' is the weighted mean of 'neighbouring' wave components in the original spectrum. Conversely each wave component in the original spectrum contributes to 'neighbouring' components of the new spectrum. From equations (65) and (79), the number of wave-lengths of any wave component intercepted on the x-axis inside S is u, and the corresponding number on the y-axis is v. The width of the spread pattern in (81) is of order unity. Hence, for this purpose, 'neighbouring' wave components are those such that the number of

wave-lengths intercepted on any diameter of S does not differ by more than 2 or 3 from the corresponding number for the original wave component.

The replacement of the 'sharp' spectrum A by the 'blurred' spectrum A' may be considered as the result of our inability to define the spectrum exactly from a knowledge of the conditions over only a limited region. For practical purposes, however, the amount of blurring will not usually affect the frequency characteristics of **F** to a very great extent.

4. Wave motion in a heavy compressible fluid

In the present investigation the water has so far been treated as incompressible. This assumption is only valid so long as the time taken for a disturbance to be propagated to the bottom is small compared with the period of the waves, that is,

$$h/c \ll T \quad \text{or} \quad h \ll cT, \tag{83}$$

where c is the velocity of sound in water. For ocean waves h may be of the order of several kilometres, c is about 1·4 km./sec. and T lies between about 5 and 20 sec. The condition (83) is therefore no longer satisfied. It follows that in practice the compressibility of the water must be taken into account.

Surface waves in a heavy compressible fluid were first considered by Pidduck (1910, 1912) in connexion with the propagation of an impulse applied to the surface of the water. His method involves the neglect of squares and products of the displacements and is thus only a first-order theory. The relation obtained by him between the period and wave-length of the waves was discussed by Whipple & Lee (1935), who showed that for waves of a few seconds' period two possible types exist. On the one hand there is a motion approximating very nearly to an ordinary surface wave in incompressible fluid, in which the particle displacement decreases exponentially downwards (to the first order). This may be called a gravity-type wave. On the other hand, there are long waves controlled chiefly by the compressibility of the medium, and hardly attenuated at all with depth. These may be called compression-type waves. Stoneley (1926) and Scholte (1943) have in addition taken into account the elasticity of the sea bed. Here again the two types of wave may be distinguished.

The pressure variations of particular interest to us are, however, of the second order, and to investigate these it will be necessary to work to the second approximation. In the following we shall consider a case of special interest, namely, the motion which in the first approximation is a standing wave of gravity type. We shall find that in the second approximation long compression-type waves appear. One consequence of this is that in the second-order theory pure gravity-type or pure compression-type waves do not in general exist; the one type of wave cannot exist without the other. As a compensating advantage, however, our work leads us to the distinction of two definite regions of the fluid in one of which gravity, and in the other compressibility, is the controlling factor.

4·1. *General equations*

Take Cartesian axes (x, y, z) with the origin in the undisturbed free surface, the y-axis parallel to the wave crests, and the z-axis vertically downwards. It is assumed that the motion is periodic in the x-direction with wave-length λ. Let $z = h$ be the equation of the rigid bottom and $z = \zeta$ the equation of the free surface. Also let $\mathbf{u}$ = velocity, p = pressure,

ρ = density, and let p_s and ρ_s denote the (constant) values of p and ρ at the free surface. We shall assume that viscosity is negligible and that the velocity is irrotational, so that

$$\mathbf{u} = -\text{grad}\,\phi. \tag{84}$$

We assume also that ρ is a function of p only. Then the equations of motion may be integrated (Lamb 1932, § 20) to give

$$\frac{\partial\phi}{\partial t} - \tfrac{1}{2}\mathbf{u}^2 + gz - P = 0, \tag{85}$$

where ϕ contains an arbitrary function of the time t and where

$$P = \int_{\rho_s}^{\rho} \frac{dp}{\rho}. \tag{86}$$

We assume, lastly, as the relation connecting p and ρ,

$$\frac{dp}{d\rho} = c^2 = \text{constant}, \tag{87}$$

that is, the velocity of sound c in the medium is constant. Then from equation (86)

$$P = c^2 \int_{\rho_s}^{\rho} \frac{dp}{\rho} = c^2 \log(\rho/\rho_s). \tag{88}$$

Now the equation of continuity may be written

$$\frac{D\rho}{Dt} - \rho\nabla^2\phi = 0, \tag{89}$$

where D/Dt denotes differentiation following the motion. Hence

$$\nabla^2\phi = \frac{1}{\rho}\frac{D\rho}{Dt} = \frac{D}{Dt}(\log\rho), \tag{90}$$

and so from (88)

$$\nabla^2\phi = \frac{1}{c^2}\frac{DP}{Dt}. \tag{91}$$

On eliminating P between equations (85) and (91) we have

$$\begin{aligned} c^2\nabla^2\phi &= \frac{D}{Dt}\left(\frac{\partial\phi}{\partial t} - \tfrac{1}{2}\mathbf{u}^2 + gz\right) \\ &= \frac{\partial^2\phi}{\partial t^2} - \frac{\partial}{\partial t}(\tfrac{1}{2}\mathbf{u}^2) + \mathbf{u}.\text{grad}\,\frac{\partial\phi}{\partial t} - \mathbf{u}.\text{grad}\,(\tfrac{1}{2}\mathbf{u}^2) - g\frac{\partial\phi}{\partial z}. \end{aligned} \tag{92}$$

But

$$\mathbf{u}.\text{grad}\,\frac{\partial\phi}{\partial t} = \mathbf{u}.\frac{\partial}{\partial t}(\text{grad}\,\phi) = -\frac{\partial}{\partial t}(\tfrac{1}{2}\mathbf{u}^2) \tag{93}$$

Hence

$$\frac{\partial^2\phi}{\partial t^2} - c^2\nabla^2\phi - g\frac{\partial\phi}{\partial z} - \frac{\partial}{\partial t}(\tfrac{1}{2}\mathbf{u}^2) - \mathbf{u}.\text{grad}\,(\tfrac{1}{2}\mathbf{u}^2) = 0. \tag{94}$$

This is our differential equation for ϕ. We consider now the conditions to be satisfied at the boundaries.

The boundary condition in the plane $z = h$ is simply

$$\left(\frac{\partial\phi}{\partial z}\right)_{z=h} = 0. \tag{95}$$

At the free surface $z = \zeta$ we have $p = p_s$, and therefore

$$P_{z=\zeta} = 0. \tag{96}$$

Thus from equation (85)
$$\left(\frac{\partial\phi}{\partial t} - \tfrac{1}{2}\mathbf{u}^2 + gz\right)_{z=\zeta} = 0. \tag{97}$$

Since a particle in the free surface always remains in the free surface we have also

$$\left(\frac{DP}{Dt}\right)_{=\zeta} = 0, \tag{98}$$

and so from (91)
$$(\nabla^2\phi)_{z=\zeta} = 0. \tag{99}$$

Equations (97) and (99) are to be satisfied at the surface $z = \zeta$. It is more convenient, however, to replace these by conditions to be satisfied in the plane $z = 0$. This may be done by expanding the equations in a Taylor series as follows:

$$\left(\frac{\partial\phi}{\partial t} - \tfrac{1}{2}\mathbf{u}^2\right)_{z=0} + \zeta\left(\frac{\partial^2\phi}{\partial t\,\partial z} - \mathbf{u}\cdot\frac{\partial\mathbf{u}}{\partial z} + g\right)_{z=0} + \ldots = 0 \tag{100}$$

and
$$(\nabla^2\phi)_{z=0} + \zeta\left(\frac{\partial}{\partial z}\nabla^2\phi\right)_{z=0} + \ldots = 0. \tag{101}$$

In order to define the solution completely it is necessary to add a further condition derived from the assumption that the origin is in the undisturbed free surface. Since the mass contained below the free surface is the same as in the undisturbed state we have

$$\int_0^\lambda dx \int_\zeta^h \rho\, dz = \int_0^\lambda dx \int_0^h \rho_0\, dz, \tag{102}$$

where a suffix 0 denotes the value in the undisturbed state. Equation (102) may be written

$$\int_0^\lambda dx \int_0^h (\rho - \rho_0)\, dz - \int_0^\lambda dx \int_0^\zeta \rho\, dz = 0. \tag{103}$$

In the second term let ρ be expanded in a Taylor series from $z = 0$. After integrating with respect to z we have

$$\int_0^\lambda dx \int_0^h (\rho - \rho_0)\, dz - \int_0^\lambda dx\left[\zeta\rho_{z=0} + \tfrac{1}{2}\zeta^2\left(\frac{\partial\rho}{\partial z}\right)_{z=0} + \ldots\right] = 0. \tag{104}$$

From equations (85) and (88), ρ is given in terms of ϕ by

$$\rho/\rho_s = e^{P/c^2} = \mathrm{e}^{(\partial\phi/\partial t - \frac{1}{2}\mathbf{u}^2 + gz)/c^2} \tag{105}$$

so that
$$\rho_0/\rho_s = \mathrm{e}^{gz/c^2}. \tag{106}$$

We also have, from (87),

$$p - p_s = c^2(\rho - \rho_s), \tag{107}$$

$$p_0 - p_s = c^2\rho_s\,\mathrm{e}^{gz/c^2 - 1}. \tag{108}$$

We seek solutions for ϕ by a method of successive approximations. Let

$$\left.\begin{aligned} \phi &= \epsilon\phi_1 + \epsilon^2\phi_2 + \ldots, \\ \mathbf{u} &= \epsilon\mathbf{u}_1 + \epsilon^2\mathbf{u}_2 + \ldots, \\ \zeta &= \epsilon\zeta_1 + \epsilon^2\zeta_2 + \ldots, \\ p - p_0 &= \epsilon p_1 + \epsilon^2 p_2 + \ldots, \\ \rho - \rho_0 &= \epsilon\rho_1 + \epsilon^2\rho_2 + \ldots, \end{aligned}\right\} \tag{109}$$

where ϵ is a small parameter. On substituting in equations (94), (95) and (101) and equating coefficients of the first power of ϵ we have

$$\left.\begin{aligned} &\frac{\partial^2\phi_1}{\partial t^2}-c^2\nabla^2\phi_1-g\frac{\partial\phi_1}{\partial z}=0,\\ &\left(\frac{\partial\phi_1}{\partial z}\right)_{z=h}=0,\\ &(\nabla^2\phi_1)_{z=0}=0, \end{aligned}\right\} \tag{110}$$

and from equations (84), (100), (105) and (107)

$$\left.\begin{aligned} &\mathbf{u}_1=-\operatorname{grad}\phi_1,\\ &g\zeta_1=-\left(\frac{\partial\phi_1}{\partial t}\right)_{z=0},\\ &p_1/\rho_s=c^2\rho_1/\rho_s=\frac{\partial\phi_1}{\partial t}\,\mathrm{e}^{2\gamma z}, \end{aligned}\right\} \tag{111}$$

where $\gamma=g/2c^2$. Similarly for the second approximation we find

$$\left.\begin{aligned} &\frac{\partial^2\phi_2}{\partial t^2}-c^2\nabla^2\phi_2-g\frac{\partial\phi_2}{\partial z}=\frac{\partial}{\partial t}(\mathbf{u}_1^2),\\ &\left(\frac{\partial\phi_2}{\partial z}\right)_{z=h}=0,\\ &(\nabla^2\phi_2)_{z=0}=-\zeta_1\left(\frac{\partial}{\partial z}\nabla^2\phi_1\right)_{z=0}, \end{aligned}\right\} \tag{112}$$

and

$$\left.\begin{aligned} &\mathbf{u}_2=-\operatorname{grad}\phi_2,\\ &g\zeta_2=-\left(\frac{\partial\phi_2}{\partial t}-\tfrac{1}{2}\mathbf{u}_1^2\right)_{z=0}-\zeta_1\left(\frac{\partial^2\phi_1}{\partial z\,\partial t}\right)_{z=0},\\ &p_2/\rho_s=c^2\rho_2/\rho_s=\left[\frac{\partial\phi_2}{\partial t}-\tfrac{1}{2}\mathbf{u}_1^2+\frac{1}{2c^2}\left(\frac{\partial\phi_1}{\partial t}\right)^2\right]\mathrm{e}^{2\gamma z}. \end{aligned}\right\} \tag{113}$$

On substituting for ρ and ζ in equation (104) and equating coefficients of ϵ and ϵ^2 we obtain the further conditions on ϕ_1 and ϕ_2

$$2\gamma\int_0^\lambda dx\int_0^h dz\,\frac{\partial\phi_1}{\partial t}\,\mathrm{e}^{2\gamma z}+\int_0^\lambda dx\left(\frac{\partial\phi_1}{\partial t}\right)_{z=0}=0 \tag{114}$$

and

$$\begin{aligned} &2\gamma\int_0^\lambda dx\int_0^h dz\,\frac{\partial\phi_2}{\partial t}\,\mathrm{e}^{2\gamma z}+\int_0^\lambda dx\left(\frac{\partial\phi_2}{\partial t}\right)_{z=0}\\ &\quad=2\gamma\int_0^\lambda dx\int_0^h dz\left[\tfrac{1}{2}\mathbf{u}_1^2-\frac{1}{2c^2}\left(\frac{\partial\phi_1}{\partial t}\right)^2\right]\mathrm{e}^{2\gamma z}+\int_0^\lambda dx\left[\tfrac{1}{2}\mathbf{u}_1^2+\frac{1}{g}\frac{\partial\phi_1}{\partial t}\frac{\partial^2\phi_1}{\partial z\,\partial t}-\frac{1}{2c^2}\left(\frac{\partial\phi_1}{\partial t}\right)^2\right]_{z=0}. \end{aligned} \tag{115}$$

Suppose that ϕ and ζ are any periodic functions satisfying equations (94), (95), (100) and (101). If P and ρ are defined by (105) then these equations imply also (89), (96) and (98). Provided grad P is not identically zero, (96) and (98) show that $z=\zeta$ is a surface moving with the fluid. But since the equation of continuity (89) is satisfied, it follows that the left-hand side of (102), (103) or (104) is at most a constant. Hence any periodic solution $\phi_1=\phi_1^*$ of equations (110) must make the left-hand side of equation (114) a constant, say C_1^*. Then a solution of (114) is given by

$$\phi_1=\phi_1^*-C_1^*\,\mathrm{e}^{-2\gamma h}\,t. \tag{116}$$

But this also satisfies equations (110). Hence if ϕ_1^* is any periodic solution of (110) a solution of all four equations (110) and (114) may be found by adding to ϕ_1^* a constant multiple of t (that is, by increasing the pressure uniformly). Similarly if ϕ_2^* is any periodic solution of (112) a solution of all four equations (112) and (115) may be found by adding to ϕ_2^* a constant multiple of t. These results may be verified directly by differentiating equations (114) and (115) with respect to t and using equations (110) and (112).

4·2. *First approximation and period equation*

Let us assume for ϕ_1 a simple progressive wave of the form

$$\phi_1 = Z(z)\, e^{i(kx+\sigma t)}, \tag{117}$$

where $k = 2\pi/\lambda$, $\sigma = 2\pi/T$ and Z is a function of z only. Writing

$$Z = e^{-\gamma z} Z_1(z), \tag{118}$$

and substituting in the first of equations (110) we find

$$\frac{d^2 Z_1}{dz^2} - \alpha^2 Z_1 = 0, \tag{119}$$

where
$$\alpha^2 = k^2 - \sigma^2/c^2 + \gamma^2. \tag{120}$$

Assuming $\alpha \neq 0$ we have
$$Z_1 = A e^{\alpha z} + B e^{-\alpha z}, \tag{121}$$

where A and B are constants, and hence

$$\phi_1 = [A e^{-(\gamma-\alpha)z} + B e^{-(\gamma+\alpha)z}]\, e^{i(kx+\sigma t)}. \tag{122}$$

From the last two of equations (110) we have two simultaneous equations for A and B:

$$\left.\begin{aligned} -(\gamma-\alpha)\, e^{-(\gamma-\alpha)h} A - (\gamma+\alpha)\, e^{-(\gamma+\alpha)h} B = 0, \\ \{(\gamma-\alpha)^2 - k^2\} A + \{(\gamma+\alpha)^2 - k^2\} B = 0. \end{aligned}\right\} \tag{123}$$

Let $\Delta(\sigma, k)$ denote the determinant of these equations, so that

$$\begin{aligned} \Delta(\sigma,k) &= -(\gamma-\alpha)\{(\gamma+\alpha)^2-k^2\} e^{-(\gamma-\alpha)h} + (\gamma+\alpha)\{(\gamma-\alpha)^2-k^2\} e^{-(\gamma+\alpha)h} \\ &= -2 e^{-\gamma h}[\gamma(\gamma^2-\alpha^2-k^2)\sinh \alpha h + \alpha(\gamma^2-\alpha^2+k^2)\cosh \alpha h]. \end{aligned} \tag{124}$$

In order that non-zero solutions of (123) may exist, $\Delta(\sigma, k)$ must vanish, giving

$$f(\alpha h) \equiv \alpha h \coth \alpha h - P(\alpha h)^2 - Q = 0, \tag{125}$$

where
$$P = \frac{g}{h\sigma^2}, \quad Q = \gamma h(1 - P\gamma h). \tag{126}$$

If σ and h are given, (125) is an equation for determining α and hence k and λ. When αh tends to zero, f tends to the finite value $(1-Q)$, which will be assumed to be positive. When αh is large and positive $f(\alpha h)$ is negative. But, writing $\eta = \alpha^2 h^2$, we may easily show that $d^2 f/d\eta^2$ is always negative when α is real, so that f has only one positive zero, which corresponds to a wave of gravity type. There are an infinity of imaginary zeros, each corresponding to a wave of compression type (Whipple & Lee 1935). It may also be shown that $f(\alpha h)$ has no complex zeroes.

We shall now assume that α is the positive real root of equation (125). Since $f(\gamma h)$ is positive it follows that

$$\gamma^2 < \alpha^2, \quad k^2 > \sigma^2/c^2 > 0, \tag{127}$$

so that the corresponding value of k is real. Then from equations (123) we have

$$\phi_1 = [(\gamma+\alpha)\,\mathrm{e}^{-\alpha h-(\gamma-\alpha)z} - (\gamma-\alpha)\,\mathrm{e}^{\alpha h-(\gamma+\alpha)z}]\,\mathrm{e}^{i(kx+\sigma t)}. \tag{128}$$

This solution also satisfies equation (114). Since the equations for the first approximation are all linear the sum of any number of solutions is also a solution. We may therefore take as our first approximation

$$\phi_1 = [(\gamma+\alpha)\,\mathrm{e}^{-\alpha h-(\gamma-\alpha)z} - (\gamma-\alpha)\,\mathrm{e}^{\alpha h-(\gamma+\alpha)z}]\,[b_1\sin(kx-\sigma t)+b_2\sin(kx+\sigma t)], \tag{129}$$

representing two waves of the same wave-length travelling in opposite directions.

4·3. *Second approximation*

After substituting in equations (112) and (115) and simplifying we find the following equations for ϕ_2:

$$\begin{aligned}\frac{\partial^2\phi_2}{\partial t^2} - c^2\nabla^2\phi_2 - g\frac{\partial\phi_2}{\partial z} &= [C^{(2)}\,\mathrm{e}^{-2(\gamma-\alpha)z} + C^{(2)}\,\mathrm{e}^{-2(\gamma+\alpha)z} - 2C^{(3)}\,\mathrm{e}^{-2\gamma z}]\,[b_1^2\sin 2(kx-\sigma t) - b_2^2\sin 2(kx+\sigma t)]\\ &\quad + [C^{(4)}\,\mathrm{e}^{-2(\gamma-\alpha)z} + C^{(5)}\,\mathrm{e}^{-2(\gamma+\alpha)z} - 2C^{(6)}\,\mathrm{e}^{-2\gamma z}]\,2b_1b_2\sin 2\sigma t,\end{aligned} \tag{130}$$

$$\left(\frac{\partial\phi_2}{\partial z}\right)_{z=h} = 0, \tag{131}$$

$$(\nabla^2\phi_2)_{z=0} = D[b_1^2\sin 2(kx-\sigma t) - b_2^2\sin 2(kx+\sigma t) + 2b_1b_2\sin 2\sigma t], \tag{132}$$

$$2\gamma\int_0^\lambda dx\int_0^h dz\,\frac{\partial\phi_2}{\partial t}\,\mathrm{e}^{2\gamma z} + \int_0^\lambda dx\left(\frac{\partial\phi_2}{\partial t}\right)_{z=0} = E^{(1)}(b_1^2+b_2^2) + E^{(2)}2b_1b_2\cos 2\sigma t, \tag{133}$$

where $C^{(1)}$, $C^{(2)}$, ..., $C^{(6)}$, D and $E^{(1)}$ are constants given by

$$\left.\begin{aligned} C^{(1)} &= -\sigma\{(\gamma-\alpha)^2-k^2\}\,(\gamma+\alpha)^2\,\mathrm{e}^{-2\alpha h}, & C^{(4)} &= -\sigma\{(\gamma-\alpha)^2+k^2\}\,(\gamma+\alpha)^2\,\mathrm{e}^{-2\alpha h},\\ C^{(2)} &= -\sigma\{(\gamma+\alpha)^2-k^2\}\,(\gamma-\alpha)^2\,\mathrm{e}^{2\alpha h}, & C^{(5)} &= -\sigma\{(\gamma+\alpha)^2+k^2\}\,(\gamma-\alpha)^2\,\mathrm{e}^{2\alpha h},\\ C^{(3)} &= -\sigma\{\gamma^2-\alpha^2-k^2\}\,(\gamma^2-\alpha^2), & C^{(6)} &= -\sigma\{\gamma^2-\alpha^2+k^2\}\,(\gamma^2-\alpha^2),\end{aligned}\right\} \tag{134}$$

and

$$D = -\frac{4\sigma}{g}\gamma\alpha^2(\gamma^2-\alpha^2), \quad E^{(1)} = -\lambda\alpha^2(\gamma^2-\alpha^2) \tag{135}$$

(the value of $E^{(2)}$ will not be required). We first eliminate the right-hand side of equation (130) by the substitution

$$\begin{aligned}\phi_2 &= [F^{(1)}\,\mathrm{e}^{-2(\gamma-\alpha)z} + F^{(2)}\,\mathrm{e}^{-2(\gamma+\alpha)z} - 2F^{(3)}\,\mathrm{e}^{-2\gamma z}]\,[b_1^2\sin 2(kx-\sigma t) - b_2^2\sin 2(kx+\sigma t)]\\ &\quad + [F^{(4)}\,\mathrm{e}^{-2(\gamma-\alpha)z} + F^{(5)}\,\mathrm{e}^{-2(\gamma+\alpha)z} - 2F^{(6)}\,\mathrm{e}^{-2\gamma z}]\,2b_1b_2\sin 2\sigma t + \phi_2',\end{aligned} \tag{136}$$

where

$$\left.\begin{aligned} F^{(1)} &= \frac{C^{(1)}}{-4\sigma^2-4c^2\{(\gamma-\alpha)^2-k^2\}+2g(\gamma-\alpha)},\\ F^{(2)} &= \frac{C^{(2)}}{-4\sigma^2-4c^2\{(\gamma+\alpha)^2-k^2\}+2g(\gamma+\alpha)},\\ F^{(3)} &= \frac{C^{(3)}}{-4\sigma^2-4c^2(\gamma^2-k^2)+2g\gamma},\end{aligned}\right\} \tag{137}$$

and
$$\left.\begin{aligned} F^{(4)} &= \frac{C^{(4)}}{-4\sigma^2-4c^2(\gamma-\alpha)^2+2g(\gamma-\alpha)}, \\ F^{(5)} &= \frac{C^{(5)}}{-4\sigma^2-4c^2(\gamma+\alpha)^2+2g(\gamma+\alpha)}, \\ F^{(6)} &= \frac{C^{(6)}}{-4\sigma^2-4c^2\gamma^2+2g\gamma}. \end{aligned}\right\} \tag{138}$$

This gives

$$\frac{\partial^2\phi_2'}{\partial t^2}-c^2\nabla^2\phi_2'-g\frac{\partial\phi_2'}{\partial z}=0, \tag{139}$$

$$\left(\frac{\partial\phi_2'}{\partial z}\right)_{z=h}=G^{(1)}[b_1^2\sin 2(kx-\sigma t)-b_2^2\sin 2(kx+\sigma t)]+G^{(2)}2b_1b_2\sin 2\sigma t, \tag{140}$$

$$(\nabla^2\phi_2')_{z=0}=(D+H^{(1)})\,[b_1^2\sin 2(kx-\sigma t)-b_2^2\sin 2(kx+\sigma t)]+(D+H^{(2)})\,2b_1b_2\sin 2\sigma t, \tag{141}$$

$$2\gamma\int_0^\lambda dx\int_0^h dz\,\frac{\partial\phi_2'}{\partial t}\,e^{2\gamma z}+\int_0^\lambda dx\left(\frac{\partial\phi_2'}{\partial t}\right)_{z=0}=E^{(1)}(b_1^2+b_2^2)+(E^{(2)}+I)\,2b_1b_2\cos 2\sigma t, \tag{142}$$

where
$$\left.\begin{aligned} G^{(1)} &= 2(\gamma-\alpha)\,e^{-2(\gamma-\alpha)h}F^{(1)}+2(\gamma+\alpha)\,e^{-2(\gamma+\alpha)h}F^{(2)}-4\gamma\,e^{-2\gamma h}F^{(3)}, \\ G^{(2)} &= 2(\gamma-\alpha)\,e^{-2(\gamma-\alpha)h}F^{(4)}+2(\gamma+\alpha)\,e^{-2(\gamma+\alpha)h}F^{(5)}-4\gamma\,e^{-2\gamma h}F^{(6)}, \end{aligned}\right\} \tag{143}$$

and
$$\left.\begin{aligned} H^{(1)} &= -4\{(\gamma-\alpha)^2-k^2\}\,F^{(1)}-4\{(\gamma+\alpha)^2-k^2\}\,F^{(2)}+8(\gamma^2-k^2)\,F^{(3)}, \\ H^{(2)} &= -4(\gamma-\alpha)^2F^{(1)}-4(\gamma+\alpha)^2F^{(5)}+8\gamma^2F^{(6)}. \end{aligned}\right\} \tag{144}$$

We now write

$$\begin{aligned}\phi_2' = {} & [J^{(1)}e^{-(\gamma-\alpha')z}+J^{(2)}e^{-(\gamma+\alpha')z}]\,[b_1^2\sin 2(kx-\sigma t)-b_2^2\sin 2(kx+\sigma t)] \\ & +[J^{(3)}e^{-(\gamma-\alpha'')z}+J^{(4)}e^{-(\gamma+\alpha'')z}]\,2b_1b_2\sin 2\sigma t+\phi_2'',\end{aligned} \tag{145}$$

where
$$\alpha'^2=4k^2-4\sigma^2/c^2+\gamma^2,\quad \alpha''^2=-4\sigma^2/c^2+\gamma^2, \tag{146}$$

and $J^{(1)}$, $J^{(2)}$, $J^{(3)}$ and $J^{(4)}$ are to be chosen so as to reduce the right-hand sides of equations (140) and (141) to zero. We must have

$$\left.\begin{aligned} -(\gamma-\alpha')\,e^{-(\gamma-\alpha')h}J^{(1)}-(\gamma+\alpha')\,e^{-(\gamma+\alpha')h}J^{(2)} &= G^{(1)}, \\ \{(\gamma-\alpha')^2-4k^2\}\,J^{(1)}+\{(\gamma+\alpha')^2-4k^2\}\,J^{(2)} &= D+H^{(1)}, \end{aligned}\right\} \tag{147}$$

and
$$\left.\begin{aligned} -(\gamma-\alpha'')\,e^{-(\gamma-\alpha'')h}J^{(3)}-(\gamma+\alpha'')\,e^{-(\gamma+\alpha'')h}J^{(4)} &= G^{(2)}, \\ (\gamma-\alpha'')^2J^{(3)}\qquad\qquad +(\gamma+\alpha'')^2J^{(4)} &= D+H^{(2)}, \end{aligned}\right\} \tag{148}$$

giving
$$\left.\begin{aligned} J^{(1)} &= \frac{\{(\gamma+\alpha')^2-4k^2\}\,G^{(1)}+(\gamma+\alpha')\,e^{-(\gamma+\alpha')h}\,(D+H^{(1)})}{\Delta(2\sigma,2k)}, \\ J^{(2)} &= -\frac{\{(\gamma-\alpha')^2-4k^2\}\,G^{(1)}+(\gamma-\alpha')\,e^{-(\gamma-\alpha')h}\,(D+H^{(1)})}{\Delta(2\sigma,2k)}, \end{aligned}\right\} \tag{149}$$

and
$$\left.\begin{aligned} J^{(3)} &= \frac{(\gamma+\alpha'')^2\,G^{(2)}+(\gamma+\alpha'')\,e^{-(\gamma+\alpha'')h}\,(D+H^{(2)})}{\Delta(2\sigma,0)}, \\ J^{(4)} &= -\frac{(\gamma-\alpha'')^2\,G^{(2)}+(\gamma-\alpha'')\,e^{-(\gamma-\alpha'')h}\,(D+H^{(2)})}{\Delta(2\sigma,0)}, \end{aligned}\right\} \tag{150}$$

where
$$\left.\begin{aligned}\Delta(2\sigma, 2k) &= -2\,\mathrm{e}^{-\gamma h}[\gamma(\gamma^2-\alpha'^2-4k^2)\sinh\alpha' h+\alpha'(\gamma^2-\alpha'^2)\cosh\alpha' h],\\ \Delta(2\sigma, 0) &= -2\,\mathrm{e}^{-\gamma h}[\gamma(\gamma^2-\alpha''^2)\sinh\alpha'' h+\alpha''(\gamma^2-\alpha''^2)\cosh\alpha'' h],\end{aligned}\right\} \tag{151}$$

provided neither $\Delta(2\sigma, 2k)$ nor $\Delta(2\sigma, 0)$ vanishes. Now if θ is any real number we have

$$\Delta(\theta\sigma, \theta k) = -2\,\mathrm{e}^{-\gamma h}\sinh\beta h[\gamma(1-2k^2c^2/\sigma^2)+\beta\coth\beta h]\,\theta^2\sigma^2/c^2, \tag{152}$$

where
$$\beta^2 = \theta^2(k^2-\sigma^2/c^2)+\gamma^2. \tag{153}$$

Since $(k^2-\sigma^2/c^2)$ is positive (equation (127)), β^2 is a positive, increasing function of θ^2. But $\beta\coth\beta h$ is an increasing function of β^2 when $\beta^2>0$ and hence is an increasing function of θ^2. Equation (152) then shows that $\Delta(\theta\sigma, \theta k)$ cannot vanish for more than one positive value of θ. But $\Delta(\sigma, k)$ vanishes and therefore $\Delta(2\sigma, 2k)$ cannot vanish.

It is quite possible, on the other hand, that $\Delta(2\sigma, 0)$ may be zero. The physical significance of this case will be discussed later. For the present it will be assumed that $\Delta(2\sigma, 0)$ is different from zero.

As a result of our choice of $J^{(1)}$, etc., we have for ϕ_2'' the following equations:

$$\left.\begin{aligned}&\partial^2\phi_2''-c^2\nabla^2\phi_2''-g\frac{\partial\phi_2''}{\partial z} = 0,\\ &\left(\frac{\partial\phi_2''}{\partial z}\right)_{z=h} = 0,\\ &(\nabla^2\phi_2'')_{z=0} = 0,\end{aligned}\right\} \tag{154}$$

$$2\gamma\int_0^\lambda dx\int_0^h dz\,\frac{\partial\phi_2''}{\partial t}\,\mathrm{e}^{2\gamma z}+\int_0^\lambda dx\left(\frac{\partial\phi_2''}{\partial t}\right)_{z=0} = E^{(1)}(b_1^2+b_2^2)+(E^{(2)}+I+K)\,2b_1b_2\cos 2\sigma t, \tag{155}$$

where K is a constant. Now it was shown earlier that a solution of all four equations (112) and (114) could be obtained by adding a constant multiple of t to any given solution of (112). It follows, by subtraction, that a solution of all four equations (154) and (155) may be obtained by adding a constant multiple of t to any solution of (154). But (154) are satisfied by $\phi_2'' = 0$. Hence we have

$$\phi_2'' = C''t, \tag{156}$$

where on substitution in (155) we find

$$\lambda\,\mathrm{e}^{2\gamma h}C'' = E^{(1)}(b_1^2+b_2^2). \tag{157}$$

We have incidentally shown that
$$E^{(2)}+I+K = 0. \tag{158}$$

We therefore have finally

$$\begin{aligned}\phi_2 = {}&[F^{(1)}\mathrm{e}^{2\alpha z}+F^{(2)}\mathrm{e}^{-2\alpha z}-2F^{(3)}]\,\mathrm{e}^{-2\gamma z}\,[b_1^2\sin 2(kx-\sigma t)-b_2^2\sin 2(kx+\sigma t)]\\ &+[F^{(4)}\mathrm{e}^{2\alpha z}+F^{(5)}\mathrm{e}^{-2\alpha z}-2F^{(6)}]\,\mathrm{e}^{-2\gamma z}\,2b_1b_2\sin 2\sigma t\\ &+[J^{(1)}\mathrm{e}^{\alpha' z}+J^{(2)}\mathrm{e}^{-\alpha' z}]\,\mathrm{e}^{-\gamma z}\,[b_1^2\sin 2(kx-\sigma t)-b_2^2\sin 2(kx+\sigma t)]\\ &+[J^{(3)}\mathrm{e}^{\alpha'' z}+J^{(4)}\mathrm{e}^{-\alpha'' z}]\,\mathrm{e}^{-\gamma z}\,2b_1b_2\sin 2\sigma t\\ &+E^{(1)}k^{-1}\,\mathrm{e}^{-2\gamma h}\,(b_1^2+b_2^2)\,t.\end{aligned} \tag{159}$$

4·4. *Discussion*

For ocean waves we may take

$$\begin{aligned}g &= 0{\cdot}98\times10^3\ \text{cm./sec.}, \quad & c &= 1{\cdot}4\times10^5\ \text{cm./sec.}\\ \sigma &= 0{\cdot}5\ \text{sec.}^{-1}, & h &< 10^6\ \text{cm.}\end{aligned} \tag{160}$$

This gives
$$P\gamma h = 1{\cdot}0\times10^{-4}, \quad \gamma h < 2{\cdot}5\times10^{-2}, \tag{161}$$

and so
$$Q = \gamma h(1-P\gamma h) < 2{\cdot}5\times10^{-2}. \tag{162}$$

Since $\alpha h \coth \alpha h \geqslant 1$ for all real values of α, equation (126) shows that $P\alpha h$ is of the same order as $\coth \alpha h$. Hence

$$\gamma/\alpha = P\gamma h/P\alpha h \simeq 10^{-4}. \tag{163}$$

Our method will be to evaluate the constants in equation (159) by expanding in powers of γ/α. From (126) we have

$$\coth \alpha h = P\alpha h[1+O(\gamma/\alpha)], \tag{164}$$

so that

$$\sigma^2/c^2 = 2\gamma\alpha/P\alpha h = 2\gamma\alpha \tanh \alpha h[1+O(\gamma/\alpha)] \tag{165}$$

and

$$k^2 = \alpha^2[1+2(\gamma/\alpha)\tanh \alpha h + O(\gamma/\alpha)^2]. \tag{166}$$

Hence, retaining only the terms of highest order in γ/α, we find

$$\left.\begin{aligned} F^{(1)} &= \frac{\gamma\alpha^3}{\sigma}\frac{e^{-\alpha h}\sinh \alpha h}{\cosh^2 \alpha h}, & F^{(4)} &= \frac{\gamma\alpha^3}{\sigma} e^{-2\alpha h}\tanh \alpha h, \\ F^{(2)} &= \frac{\gamma\alpha^3}{\sigma}\frac{e^{\alpha h}\sinh \alpha h}{\cosh^2 \alpha h}, & F^{(5)} &= \frac{\gamma\alpha^3}{\sigma} e^{2\alpha h}\tanh \alpha h, \\ F^{(3)} &= -\frac{\gamma\alpha^3}{\sigma}\tanh \alpha h, & F^{(6)} &= -\frac{\gamma\alpha^3}{2\sigma}\tanh \alpha h, \end{aligned}\right\} \tag{167}$$

$$\left.\begin{aligned} J^{(1)} &= -\frac{3\alpha^4}{4\sigma}\frac{e^{-\alpha' h}}{\sinh^2 \alpha h}, & J^{(3)} &= \frac{\alpha^4}{\sigma}\frac{e^{-\alpha'' h}}{\cosh \alpha'' h}\frac{\cosh 3\alpha h}{\cosh \alpha h}, \\ J^{(2)} &= -\frac{3\alpha^4}{4\sigma}\frac{e^{\alpha' h}}{\sinh^2 \alpha h}, & J^{(4)} &= \frac{\alpha^4}{\sigma}\frac{e^{\alpha'' h}}{\cosh \alpha'' h}\frac{\cosh 3\alpha h}{\cosh \alpha h}, \end{aligned}\right\} \tag{168}$$

$$E^{(1)} = \lambda\alpha^4, \quad \alpha'^2 = 4\alpha^2, \quad \alpha''^2 = -4\sigma^2/c^2. \tag{169}$$

When $b_1 b_2 \neq 0$ the first two terms in equation (159) are negligible compared with the fourth. If we also neglect quantities of order γh (though not those of order $(\gamma\alpha)^{\frac{1}{2}} h$), αh, $\alpha' h$, $\alpha'' h$ and $e^{\gamma h}$ may be replaced by kh, $2kh$, $2i\sigma h/c$ and 1 respectively, and we have

$$\sigma^2 = gk \tanh kh, \tag{170}$$

$$\phi_1 = \frac{\sigma}{k}\frac{\cosh k(z-h)}{\sinh kh}[a_1 \sin(kx-\sigma t) - a_2 \sin(kx+\sigma t)], \tag{171}$$

$$\begin{aligned} \phi_2 = &-\frac{3\sigma}{8}\frac{\cosh 2k(z-h)}{\sinh^4 kh}[a_1^2 \sin 2(kx-\sigma t) - a_2^2 \sin 2(kx+\sigma t)] \\ &-\frac{\sigma}{8}\frac{\cosh 3kh}{\sinh^2 kh \cosh kh}\frac{\cos 2\sigma(z-h)/c}{\cos 2\sigma h/c} 2a_1 a_2 \sin 2\sigma t \\ &+\frac{\sigma}{4}\frac{a_1^2+a_2^2}{\sinh^2 kh}\sigma t, \end{aligned} \tag{172}$$

where

$$a_1 = \frac{\sigma}{2k^2 \sinh kh} b_1, \quad a_2 = -\frac{\sigma}{2k^2 \sinh kh} b_2. \tag{173}$$

Let λ_g and λ_c denote the wave-lengths of a gravity wave and a compression wave respectively. Thus

$$\lambda_g = 2\pi/k, \quad \lambda_c = 2\pi c/\sigma, \quad \lambda_g/\lambda_c = (\gamma/\alpha)^{\frac{1}{2}} \tanh^{\frac{1}{2}} \alpha h. \tag{174}$$

When z is less than say $\frac{1}{2}\lambda_g$, equations (170), (171) and (172) show that the motion is independent of c and therefore unaffected by the compressibility of the water. When z is comparable with λ_g, both e^{-kz} and e^{-kh} are small, and so from equation (172) the pressure p_2 is given by

$$\frac{p_2}{\rho_s} = -2a_1 a_2 \sigma^2 \cos 2\sigma t. \tag{175}$$

Finally, when z is of the same order as λ_c, the motion reduces to the compression wave

$$\phi_2 = -\frac{\sigma \cos 2\sigma(z-h)/c}{\cos 2\sigma h/c} \sin 2\sigma t. \tag{176}$$

This wave may be regarded as being generated by the unattenuated pressure variation (175). When $\cos 2\sigma h/c$ (or more exactly $\Delta(2\sigma, 0)$) is zero, ϕ_2 becomes infinite, a situation corresponding to resonance. The necessary condition for resonance is that

$$2\sigma h/c \doteqdot (n+\tfrac{1}{2})\,\pi \quad (n = 0, 1, 2, \ldots), \tag{177}$$

that is, that the depth should be about $(\frac{1}{2}n+\frac{1}{4})$ times the length of the compression wave (176).

The ocean may therefore be divided into two regions, namely, (1) a surface layer where thickness is of order λ_g, where the motion is controlled by gravity alone and is the same as if the water as a whole were incompressible, and (2) the main part of the ocean where the motion is small and controlled only by compressibility. The distinction of two such regions is probably valid in more general types of wave motion. In equation (94) the gravity term $g\partial\phi/\partial z$ is in general small compared with the compressibility term $c^2\nabla^2\phi$. It is only near the free surface, where $\nabla^2\phi$ vanishes (equation (99)), that gravity predominates. The pressure variations at a depth λ_g, that is, in the lower part of the surface layer, are of order $\rho\sigma^2a^2$, where a is the mean amplitude at the free surface. These will produce compression waves in which the displacements are of order a^2/λ_c. But the latter will be small compared with the vertical displacement of the centre of gravity of the surface layer, which is of order a^2/λ_g, and hence will not affect the motion in the surface layer.

5. The displacement of the ground due to surface waves

In the present section we shall estimate the displacement of the ground due to a given storm at sea. Since observations are not made in the storm area itself, it is not appropriate to consider the displacement of the sea bed due to an infinite train of waves passing overhead. The storm is more correctly considered as a disturbance of finite area from which energy is propagated outwards in all directions.

The velocities of seismic waves in the sea bed being comparable with the velocity of sound in water, the general results suggested in § 4·4 are likely to remain true when the elasticity of the sea bed is also taken into account. Thus the mean pressure at a depth of say $\frac{1}{2}\lambda_g$ over any given area of the sea surface may be derived as in § 3, and the amplitude of the elastic waves may be calculated as though this pressure distribution were applied to the upper surface of the ocean. Since λ_g/λ_c is of the order of 10^{-2}, the storm area may be divided into a number of squares S whose side $2R$ is large compared with λ_g but only a fraction, say less than one-half, of the length of an elastic wave in the sea bed. Thus the amplitude of the compression waves from any given square S will be of the same order of magnitude as if the whole force were concentrated to a point at the centre of the square. The displacement from the whole storm may then be found by summing the energies from all the different squares.

5·1. *The displacement due to a concentrated force*

We take as our model an ocean of constant depth h overlying a sea bed of uniform density and elasticity. For the reasons given above, we shall be able to make use of the first-order theory of elastic waves in such a model, which was first investigated by Stoneley (1926).

The motion due to a concentrated force applied to the upper surface of the water was stated by Scholte (1943). We shall evaluate the solution rather more completely, using the method of contour integration due to Sommerfeld (1909) and Jeffreys (1926).

Let ρ_1 and ρ_2 be the densities of the water and of the sea bed, let $c = \alpha_1$ be the velocity of sound in water and α_2 and β_2 the velocities of compressional and distortional waves in the sea bed. Then if an oscillatory force $e^{i\sigma t}$ is applied to the surface of the water at the origin, the vertical displacement of the sea bed measured downwards is given by (Scholte 1943)

$$W(\sigma, r)\, e^{i\sigma t} = -\frac{1}{2\pi}\int_0^\infty \frac{J_0(\xi r)\,\xi\, d\xi}{\rho_2 \sigma^2 G(\xi)}\, e^{i\sigma t}, \tag{178}$$

where r is the horizontal distance from the origin, J_0 is Bessel's function of the first kind of zero order and $G(\xi)$ is given by

$$G(\xi) = (\beta_2/\sigma)^4 [(2\xi^2 - \sigma^2/\beta_2^2)^2 (\xi^2 - \sigma^2/\alpha_2^2)^{-\frac{1}{2}} - 4\xi^2(\xi^2 - \sigma^2/\beta_2^2)^{\frac{1}{2}}] \cosh (\xi^2 - \sigma^2/\alpha_1^2)^{\frac{1}{2}} h + (\rho_1/\rho_2)(\xi^2 - \sigma^2/\alpha_1^2)^{-\frac{1}{2}} \sinh (\xi^2 - \sigma^2/\alpha_1^2)^{\frac{1}{2}} h. \tag{179}$$

In order to ensure that the displacements at infinite depth are bounded, the signs of the radicals in equation (179) must be chosen so that the real parts of $(\xi^2 - \sigma^2/\alpha_2^2)^{\frac{1}{2}}$ and $(\xi^2 - \sigma^2/\beta_2^2)^{\frac{1}{2}}$ are positive or zero. ξ being considered as a complex variable, this restricts us initially to one sheet of the Riemann surface bounded by the cuts

$$\Re(\xi^2 - \sigma^2/\alpha_2^2)^{\frac{1}{2}} = 0, \quad \Re(\xi^2 - \sigma^2/\beta_2^2)^{\frac{1}{2}} = 0. \tag{180}$$

It will be seen that the choice of sign for $(\xi^2 - \sigma^2/\alpha_1^2)^{\frac{1}{2}}$ is immaterial, since $\cosh (\xi^2 - \sigma^2/\alpha_1^2)^{\frac{1}{2}} h$ and $(\xi^2 - \sigma^2/\alpha_1^2)^{-\frac{1}{2}} \sinh (\xi^2 - \sigma^2/\alpha_1^2)^{\frac{1}{2}} h$ are both single-valued functions of ξ, analytic at all points.

When σ is real the integral in equation (178) is indeterminate owing to the vanishing of $G(\xi)$ at certain points of the real axis. To obtain a correct interpretation we suppose σ to be complex, and take the limit as $\arg \sigma$ tends to zero. The final solution then contains converging or diverging waves according as $\arg \sigma$ tends to zero through positive or negative values. Since we require the waves to diverge we choose the latter case. Now it can be shown that, when $-\frac{1}{2}\pi < \arg \sigma < 0$, $G(\xi)$ has no zeroes in the sector $0 \leqslant \arg \xi \leqslant \frac{1}{2}\pi - \sigma$. There are therefore no zeroes on the real axis, and, in the limit when $\arg \sigma$ tends to 0, the zeroes of G approach the real axis from below. Hence the path of integration in equation (178) should be indented above the real axis near the zeroes of G (see figure 1*b*). Further, the cuts in the ξ-plane given by (180) are arcs of rectangular hyperbolas which, as $\arg \sigma$ tends to zero, approach the positive axis from below (see figure 1*a*). Hence the path of integration should be taken along the upper side of the cuts.

To evaluate the right-hand side of equation (178) we write

$$J_0(\xi r) = \tfrac{1}{2}[\mathrm{Hs}_0(\xi r) + \mathrm{Hi}_0(\xi r)] \tag{181}$$

(for the notation see Jeffreys & Jeffreys 1946, p. 544) and consider the integral in two parts. When σ is real it may be shown that G has no complex zeroes. Hence for the part involving Hs_0 the contour of integration may be deformed into the imaginary axis from 0 to $i\infty$ together with an arc of infinite radius in the first quadrant. For the part involving Hi_0 the path of integration may be deformed into (*a*) the imaginary axis from 0 to $-i\infty$, (*b*) a

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

contour Γ enclosing the cuts in the ξ-plane (see figure 1*c*), (*c*) small circles enclosing the zeroes of $G(\xi)$ in the clockwise sense and (*d*) an arc of infinite radius in the fourth quadrant. The contribution from the integrals along the imaginary axis are equal and opposite, while, since (Jeffreys & Jeffreys 1946)

$$\mathrm{Hs}_0(z) \sim \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} \mathrm{e}^{i(z-\frac{1}{4}\pi)}, \quad \mathrm{Hi}_0(z) \sim \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} \mathrm{e}^{-i(z-\frac{1}{4}\pi)}, \tag{182}$$

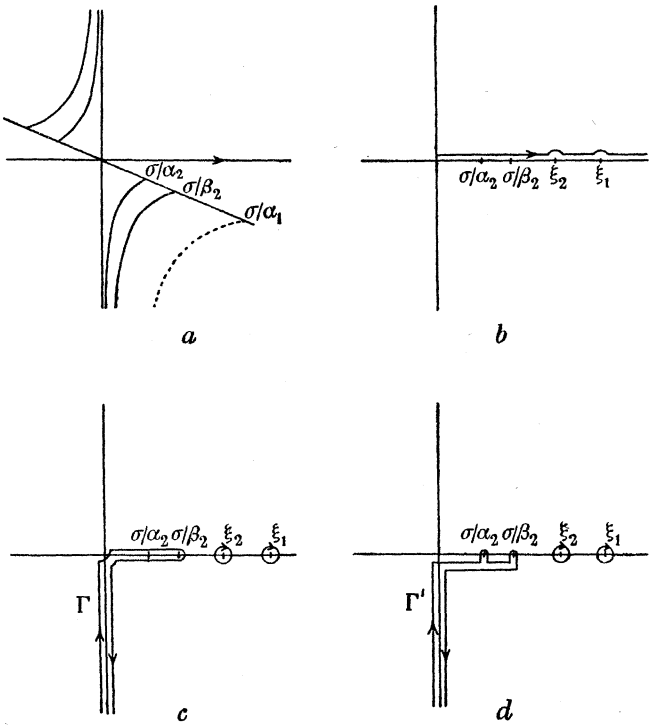

FIGURE 1. Contours of integration in the ξ-plane.

for large $|z|$ and $-\pi+\epsilon \leqslant \arg z \leqslant \pi-\epsilon$, the integrals along the two infinite arcs tend to zero. By slightly deforming the contour Γ as in figure 1*d*, it is easily shown that the contribution from this part of the integral diminishes at least as rapidly as $r^{-\frac{5}{2}}$ when r is large. Hence the main contribution comes from the neighbourhood of the zeroes, being $-2\pi i$ times the sum of the residues of the integrand there. On replacing Hi_0 by its asymptotic formula (182) we find

$$W(\sigma, r)\, \mathrm{e}^{i\sigma t} \sim \frac{\sigma^{\frac{1}{2}}}{\rho_2 \beta_2^{5/2} (2\pi r)^{\frac{1}{2}}} \sum_{m=1}^{N} c_m \mathrm{e}^{i[\sigma t - \xi_m r + (m+\frac{1}{4})\pi]}, \tag{183}$$

where

$$c_m = (-)^m \frac{(\beta_2/\sigma)^{5/2}\, \xi_m^{1/2}}{dG(\xi_m)/d\xi_m}, \tag{184}$$

and $\xi_1, \xi_2, \ldots, \xi_N$ denote the positive zeroes of $G(\xi)$ in descending order of magnitude. It can be shown that when $\alpha_1 < \beta_2$ all the zeroes are greater than σ/β_2. The zeroes of $G(\xi)$ separate alternately the zeroes of $\cosh(\xi^2 - \sigma^2/\alpha_1^2)^{\frac{1}{2}} h$, and if the latter function has n zeroes in the interval $\sigma/\beta_2 < \xi < \infty$, then N equals either n or $(n+1)$. When $\sigma h/\beta_2$ is small there is just one zero ξ_1.

Each term in equation (183) represents a diverging wave of length $2\pi/\xi_m$ and of amplitude proportional to c_m. In figure 2 c_1, c_2, c_3 and c_4 are plotted against $\sigma h/\beta_2$ for the following constants:

$$\left.\begin{aligned} \rho_1 &= 1{\cdot}0\,\text{g./cm.}^3, \quad \alpha_1 = 1{\cdot}4\,\text{km./sec.},\\ \rho_2 &= 2{\cdot}5\,\text{g./cm.}^3, \quad \beta_2 = 2{\cdot}8\,\text{km./sec.} \end{aligned}\right\} \tag{185}$$

and with Poisson's hypothesis $\alpha_2 = \sqrt{3}\beta_2$. The corresponding values of ξ_1, ξ_2, ξ_3 and ξ_4 are given in table 1. It will be seen that c_1 increases rapidly to a maximum at about $\sigma h/\beta_2 = 0{\cdot}85$ before falling away finally to zero. This maximum value occurs when the depth is about 0·27 times the wave-length of a compression wave in water, and may be interpreted as the

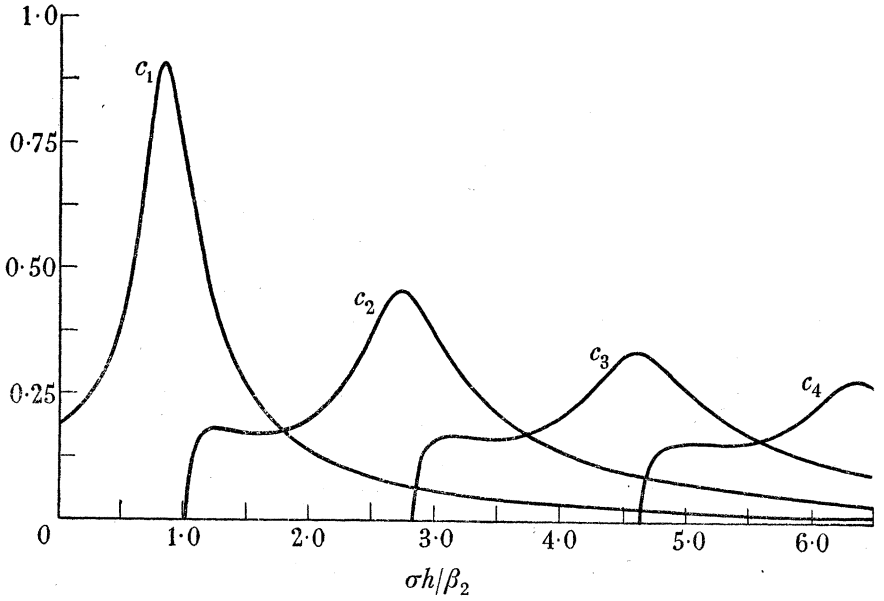

FIGURE 2. The amplitude of the vertical displacement of the sea bed as a function of the depth h.

effect of resonance. The amplitude does not, however, become infinite owing to the propagation of energy away from the source of the disturbance. c_2, c_3 and c_4 show similar resonance peaks when $\sigma h/\beta_2 = 2{\cdot}7$, 4·1 and 6·3 respectively. There are also maxima in the earlier parts of each curve. This might be expected from the fact that the group-velocity curve has two stationary values (Press & Ewing 1948). These do not, however, coincide exactly with the maxima in figure 2.

We define $\overline{W}^2$ to be the sum of the squared moduli of the terms in equation (183). Thus

$$\overline{W} = \frac{\sigma^{\frac{1}{2}}}{\rho_2\beta^{5/2}(2\pi r)^{\frac{1}{2}}}\left[\sum_{m=1}^{N} c_m^2\right]^{\frac{1}{2}}. \tag{186}$$

5·2. *The displacement of the ground in terms of the frequency spectrum of the waves*

From equation (82) we see that the wave motion in any given square S will cause a vertical displacement δ' of the ground given by

$$\delta' = -\Re 4\rho(\pi/k)^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A'A'_{-}\sigma^2 W(2\sigma, r)\, e^{2i\sigma t}\, du\, dv, \tag{187}$$

where r is the distance from the centre of the square and $W(\sigma, r)$ is given by (183). We shall now find an expression for the order of magnitude of the right-hand side of equation (187).

TABLE 1

(*a*)

$\xi_1\beta_2/\sigma$	$\sigma h/\beta_2$	c_1	$\xi_1\beta_2/\sigma$	$\sigma h/\beta_2$	c_1	$\xi_1\beta_2/\sigma$	$\sigma h/\beta_2$	c_1
1·0877	0·00	0·191	1·3784	0·89	0·890	1·8439	1·68	0·205
1·0954	0·10	0·206	1·4142	0·92	0·857	1·8974	1·99	0·139
1·1402	0·48	0·368	1·4832	0·99	0·759	1·9494	2·59	0·078
1·1832	0·63	0·565	1·5492	1·06	0·649	1·9748	3·23	0·049
1·2247	0·72	0·728	1·6125	1·13	0·542	1·9875	3·87	0·034
1·2649	0·77	0·837	1·6733	1·22	0·444	1·9975	4·87	0·021
1·3038	0·82	0·894	1·7321	1·33	0·355	2·0000	5·31	0·017
1·3416	0·85	0·908	1·7889	1·48	0·276	2·0025	5·92	0·014

(*b*)

$\xi_2\beta_2/\sigma$	$\sigma h/\beta_2$	c_2	$\xi_2\beta_2/\sigma$	$\sigma h/\beta_2$	c_2	$\xi_2\beta_2/\sigma$	$\sigma h/\beta_2$	c_2
1·0000	1·01	0·000	1·0677	1·58	0·170	1·3784	3·06	0·351
1·0005	1·03	0·038	1·0770	1·72	0·172	1·4142	3·14	0·316
1·0025	1·04	0·076	1·0863	1·86	0·180	1·4832	3·33	0·256
1·0050	1·06	0·108	1·0954	1·98	0·194	1·5492	3·54	0·206
1·0100	1·09	0·141	1·1402	2·39	0·318	1·6125	3·79	0·165
1·0198	1·14	0·168	1·1832	2·58	0·418	1·6733	4·09	0·131
1·0296	1·20	0·178	1·2247	2·70	0·454	1·7321	4·47	0·101
1·0392	1·28	0·180	1·2649	2·80	0·448	1·7889	4·99	0·076
1·0488	1·36	0·177	1·3038	2·89	0·421	1·8439	5·73	0·054
1·0583	1·46	0·173	1·3416	2·97	0·386	1·8974	6·96	0·034

(*c*)

$\xi_3\beta_2/\sigma$	$\sigma h/\beta_2$	c_3	$\xi_3\beta_2/\sigma$	$\sigma h/\beta_2$	c_3	$\xi_3\beta_2/\sigma$	$\sigma h/\beta_2$	c_3
1·0000	2·83	0·000	1·0488	3·21	0·165	1·2247	4·69	0·330
1·0005	2·84	0·036	1·0583	3·32	0·163	1·2649	4·83	0·305
1·0025	2·86	0·070	1·0677	3·44	0·162	1·3038	4·96	0·275
1·0050	2·88	0·098	1·0770	3·58	0·164	1·3416	5·09	0·245
1·0100	2·91	0·126	1·0863	3·73	0·171	1·3784	5·22	0·218
1·0198	2·97	0·151	1·0954	3·86	0·184	1·4142	5·36	0·194
1·0296	3·04	0·163	1·1402	4·30	0·280	1·4832	5·67	0·154
1·0392	3·11	0·166	1·1832	4·53	0·331	1·5492	6·02	0·123

(*d*)

$\xi_4\beta_2/\sigma$	$\sigma h/\beta_2$	c_4	$\xi_4\beta_2/\sigma$	$\sigma h/\beta_2$	c_4	$\xi_4\beta_2/\sigma$	$\sigma h/\beta_2$	c_4
1·0000	4·64	0·000	1·0198	4·80	0·138	1·0677	5·30	0·154
1·0005	4·65	0·031	1·0296	4·87	0·150	1·0770	5·44	0·157
1·0025	4·67	0·065	1·0392	4·95	0·155	1·0863	5·60	0·164
1·0050	4·69	0·090	1·0488	5·05	0·156	1·0954	5·73	0·175
1·0100	4·73	0·115	1·0583	5·17	0·155	1·1402	6·21	0·250

From the definition given in § 3·3, $A'(a, v; t_1)$ is the frequency spectrum of the hypothetical free motion in which, at time $t = t_1$, ζ and $\partial\zeta/\partial t$ take their actual values within S but are zero outside. When $t = t_1$ all the potential energy and nearly all the kinetic energy are contained inside S. Hence the total energy in the square is given by

$$2\rho g(\pi/k)^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A'A'^* \, du\, dv = (2\pi/k)^2 E, \tag{188}$$

where E denotes the mean energy per unit area of S. We define the mean amplitude a of the motion within S as half the height, from peak to trough, of the simple progressive wave

train having the same mean energy per unit area. The mean energy of a wave train of amplitude a being $\frac{1}{2}\rho g a^2$, we have from (188)

$$a^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A'A'^* \, du\, dv. \tag{189}$$

When considering a group of waves (see § 3·3) we suppose that all the energy is confined to a certain range of frequencies and directions characteristic of the group. This range will be very nearly the same for the 'blurred' spectrum A' as for the original spectrum A. Let Ω be the region in which the point $P(-uk, -vk)$, defining the length and direction of the wave components of the group, must lie. We also use Ω to denote the area of this region. Then the area of the corresponding region in the (u, v)-plane is Ω/k^2. Hence the root-mean-square value $\overline{A}$ of the modulus of A' is given by

$$\overline{A}^2\Omega/k^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A'A'^* \, du\, dv, \tag{190}$$

or from equation (188)

$$\overline{A} = ak/\Omega^{\frac{1}{2}}. \tag{191}$$

The case of most practical importance is when the motion consists of two distinct wave groups, say A'_1 and A'_2. We denote the mean amplitudes of these groups by a_1 and a_2 respectively and the corresponding areas in their frequency spectra by Ω_1 and Ω_2. The root-mean-square values of A'_1 and A'_2 are given by

$$\overline{A}_1 = a_1 k/\Omega_1^{\frac{1}{2}}, \quad \overline{A}_2 = a_2 k/\Omega_2^{\frac{1}{2}}. \tag{192}$$

On writing $A' = A'_1 + A'_2$ in equation (187) we have

$$\delta' = -\Re 4\rho(\pi/k)^2 \iint_{\Omega_1+\Omega_2} (A'_1 + A'_2)\,(A'_{1-} + A'_{2-})\,\sigma^2 W(2\sigma, r)\, e^{2i\sigma t}\, du\, dv, \tag{193}$$

where A'_{1-}, etc., is written briefly for $A'_1(-u, -v)$. Since Ω_1 defines a progressive wave group, it will contain no opposite pair of wave components, nor, similarly, will Ω_2. Thus equation (193) reduces to

$$\delta'\, e^{-2i\sigma_{12}t} = -\Re 8\rho(\pi/k)^2 \iint_{\Omega_{12}} A'_1 A'_{2-}\,\sigma^2 W(2\sigma, r)\, e^{2i(\sigma-\sigma_{12})t}\, du\, dv, \tag{194}$$

where Ω_{12} denotes the region common to Ω_1 and $-\Omega_2$, and we have introduced σ_{12}, the mean value of σ over Ω_{12}.

Now it may be assumed that there is no correlation between the phases of wave components at different points in the original spectrum A. The same will in general be true for the modified spectrum A', but because of the 'blurring' function (equation (81)) there may be some correlation for points that are close together in the (u, v)-plane. The degree of correlation will depend on the separation of the points concerned relative to the width of the blurring function, which we have seen is of order unity. Values of $A(u, v)$ much closer than this will be highly correlated, while those much more widely separated will be hardly correlated at all. Suppose then that the range of integration in (194) is divided into unit squares and the integration carried out over each square separately. The final result will be the sum of Ω_{12}/k^2 vectors of random phase and each of the order of magnitude of

$$8\rho(\pi/k)^2 \overline{A}_1 \overline{A}_2 \sigma_{12}^2 W(2\sigma_{12}, r). \tag{195}$$

Hence the order of magnitude of δ' is given by

$$\delta' \simeq 8\rho(\pi/k)^2 \bar{A}_1 \bar{A}_2 \sigma_{12}^2 (\Omega_{12}^{\frac{1}{2}}/k)\ W(2\sigma_{12}, r)\ \mathrm{e}^{2i\sigma_{12}t}. \tag{196}$$

Similarly, if the total storm area is Λ there will be $\Lambda k^2/4\pi^2$ separate squares S into which the storm area is divided. Hence the amplitude δ of the displacement from the whole storm is of the order

$$\delta \simeq 8\rho(\pi/k)^2 \bar{A}_1 \bar{A}_2 \sigma_{12}^2 (\Lambda^{\frac{1}{2}} \Omega_{12}^{\frac{1}{2}}/2\pi)\ W(2\sigma_{12}, r)\ \mathrm{e}^{2i\sigma_{12}t}. \tag{197}$$

To the same order of approximation W, which may be the sum of two or more terms, may be replaced by $\overline{W}$ (equation (186)). On substituting from equations (192) we have finally

$$\delta \simeq 4\pi\rho a_1 a_2 \sigma_{12}^2 (\Lambda\Omega_{12}/\Omega_1\Omega_2)^{\frac{1}{2}}\ \overline{W}(2\sigma_{12}, r)\ \mathrm{e}^{2i\sigma_{12}t}. \tag{198}$$

As we should expect, this formula for δ is independent of the size of the squares chosen for the subdivision of the generating area Λ. It depends only on the total generating area, on the mean wave height of each group and on the areas of the corresponding two-dimensional frequency spectra, defined by Ω_1 and Ω_2. All these are quantities of which rough estimates can in practice be made. It is interesting to remark that although δ increases as the square root of the area common to Ω_1 and $-\Omega_2$, it also diminishes with the square root of Ω_1 and Ω_2. Hence, in general, the more widely the energy is distributed in the spectrum the smaller is the resulting disturbance.

5·3. *Discussion*

We proceed now to consider the application of equation (198) in some practical cases. As was first intuitively suggested by Bernard (1941*a*), the necessary condition for the generation of microseisms on the present hypothesis is the interference of groups of waves of the same wave-length travelling in opposite directions. Although not much is at present known about the generation of waves by surface winds, observation certainly suggests that a wind blowing steadily in one direction will in the course of time generate waves or swell travelling mainly in that direction, or in a direction not differing by more than 45° from it. We must therefore either look for cases in which two wind systems are in some way opposed, or else assume the possible reflexion of wave energy from a steep coast.

Bernard suggested that favourable conditions for wave interference would be found at the centre of a cyclonic depression, where waves originating on all sides of the depression might be received. It is known that in a circular depression the winds, though mainly along the isobars, have also a component inwards towards the centre, and in fact observation of sea conditions in the 'eye' of a cyclone tend to confirm this expectation. It is well known that in such regions relatively low wind velocities may be combined with high and chaotic seas such as would be characteristic of wave interference.

Suppose then that in the centre of a circular depression in the Atlantic wave energy is being received equally from all directions with a range of periods between 10 and 16 sec. The wave-length λ in deep water being given approximately by $\lambda = gT^2/2\pi$, where T is the period, we have $\lambda_1 < \lambda < \lambda_2$, where

$$\lambda_1 = 1{\cdot}54 \times 10^4 \text{ cm.}, \quad \lambda_2 = 4{\cdot}00 \times 10^4 \text{ cm.}$$

The energy in the frequency spectrum is contained in an annular region lying between the two circles having their centres at the origin and radii $2\pi/\lambda_1$ and $2\pi/\lambda_2$ respectively. This region may be divided by any diameter of the circles into two equal regions Ω_1 and Ω_2, where

$$\Omega_1 = \Omega_2 = \Omega_{12} = 2{\cdot}15 \times 10^{-7}\,\text{cm.}^{-2}.$$

Assuming $\Lambda = 1000\,\text{km.}^2$, $\sigma_{12} = 2\pi/13\,\text{sec.}^{-1}$, $a_1 = a_2 = 3\,\text{m.}$,

we find that the coefficient of $\overline{W}\,e^{2i\sigma_{12}t}$ in equation (198) is $1{\cdot}8 \times 10^{15}$ dynes. If also

$$h = 3\,\text{km.}, \quad r = 2000\,\text{km.},$$

we find $\overline{W}(2\sigma_{12}, r) = 1{\cdot}8 \times 10^{-19}$ cm./dynes, giving as the amplitude of the displacement, from peak to trough,

$$2\,|\,\delta\,| = 6{\cdot}5 \times 10^{-4}\,\text{cm.} = 6{\cdot}5\mu.$$

The above estimate shows that the theory is in agreement with observation as regards the order of magnitude of the expected ground movement. It has been assumed that the energy is uniformly distributed within the given range of frequencies. Any concentration of energy within a narrower band in the frequency range would tend in general to increase the amplitude of the microseisms. It has also been assumed that $\overline{W}$ is constant over the whole frequency range. From the chosen value of σ_{12} we have $2\sigma_{12}h/\beta_2 = 1{\cdot}03$, so that, from figure 1, $[\Sigma c_m]^{\frac{1}{2}}$ is 0·69 or about three-quarters of its maximum value. However, since $[\Sigma c_m]^{\frac{1}{2}}$ is never less that its value of 0·191 for shallow water, and increases to 0·91 within the frequency range, the mean value chosen is certainly not a serious over-estimate.

Most cyclonic depressions are themselves in movement over the ocean with a speed comparable to that of the waves. This movement may considerably increase the effective area of wave interference. For, if the velocity of the depression as a whole exceeds the group velocity of the waves, the waves generated by winds on one side of the depression and travelling in the same general direction will interfere with those generated at a later time on the other side of the depression and travelling in the opposite direction. Thus, even if the winds blew directly along the isobars and only generated waves running strictly in that direction, there would still be a 'trail' of wave interference in the wake of the depression. In general, therefore, the motion of a depression may be expected to increase the amplitude of the microseisms generated.

The amplitude of the microseisms due to coastal wave reflexion is more difficult to estimate, since less is known about the amount of energy reflected from a sloping beach. The reflected wave is usually hidden from observation by the much larger amplitude of the incoming wave, although if the crests of the reflected wave are not parallel to those of the incoming wave the former can sometimes be clearly seen. Effective interference will take place only at those parts of the coast where the shore-line is perpendicular to the direction of propagation of some components of the wave group, and the narrower the range of directions of the incoming waves, the more critically will the amount of reflexion depend upon the direction of the shore-line. The refraction of the wave crests parallel to the shore-line in shallowing water will operate in favour of effective wave interference, although the amount of refraction is small until the depth is less than about half a wave-length.

As an example consider a swell of mean amplitude $a_1 = 2\,\text{m.}$ and period 12 to 16 sec., whose direction of propagation lies within an angle of 30°. This gives $\Omega_1 = 1{\cdot}4 \times 10^{-8}\,\text{cm.}^{-2}$.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

The direction of the reflected wave energy is then also spread over an angle of 30°. Supposing, however, the shore-line to make a mean angle of 10° with the mean direction of the incoming waves, only one-third of the angle of the reflected waves overlaps that of the incoming waves. Thus $\Omega_2 = 1{\cdot}4 \times 10^{-8}\,\text{cm.}^{-2}$, $\Omega_{12} = 0{\cdot}47 \times 10^{-8}\,\text{cm.}^{-2}$. If we assume that the reflected wave extends a distance of 10 km. from the shore with a mean amplitude equal to 5 % of that of the incoming wave, and if the effective shore-line is 600 km. in length, we have $\Lambda = 6000\,\text{sq.km.}$, $a_2 = 0{\cdot}1\,\text{m.}$ Taking $h = 0$, $r = 1000\,\text{km.}$, we find from (198) that $2\,|\,\delta\,| = 0{\cdot}3\mu$. This amplitude is rather smaller than that in the case considered previously. We conclude that the largest microseisms are probably due to wave interference in mid-ocean, although coastal reflexion may be a more common cause of microseisms of smaller amplitude. Exceptions may occur for stations near to the coast.

It has been seen that the microseism amplitudes may be increased by a factor of the order of 5 owing to the greater response of the physical system for certain depths of water. In practice, with an ocean of non-uniform depth, the amplitude will be affected by the depth of water at all points between the generating area and the observing station. Since, however, the energy density is greatest near the source of the disturbance, the depth of water in the generating area itself may be expected to be of most importance.

In so far as the sea waves must be considered to possess not a single frequency but a spectrum of finite width, we may expect that the unequal response of the ocean will cause an apparent shift of the spectrum towards those frequencies for which the response is a maximum. In the case of disturbances due to coastal reflexion, which in most instances would take place in shallow water, less frequency shift is to be expected. On the other hand, the coefficient of reflexion will very probably depend both upon the height and wave-length of the waves. There will probably also be a lengthening of the average wave period with increasing distance from the storm area, owing to the more rapid viscous damping of the higher frequencies in the spectrum. Evidence of this effect has been given by Gutenberg (1932).

6. Conclusions

Unattenuated pressure variations of the type discovered by Miche in the standing wave are a phenomenon of more general occurrence. They are due essentially to changes in the potential energy of the whole wave train. The general condition for fluctuations in the mean pressure over a wide area of the sea surface is that the frequency spectrum should contain groups of waves of the same wave-length travelling in opposite directions. The pressure fluctuations are then of twice the frequency of the corresponding waves and are proportional to the product of the wave amplitudes. Waves of compression in the ocean and sea bed should be set up, which may be of sufficient amplitude to be recorded as microseisms. For certain depths of the ocean the displacements will be increased by a factor of the order of 5 owing to resonance.

On the present theory suitable conditions of wave interference would arise near the centre of a cyclonic depression, as suggested by Bernard, but more particularly if the depression is moving rapidly. The effect of wave interference over deep water would be probably greater, under favourable conditions, than the effect of coastal wave reflexion, though the latter may be the determining factor for stations near to the coast. The periods of the microseisms

should be half those of the corresponding waves, though an apparent shift in the frequency spectrum may be expected owing to the variation of the frequency response with the depth of the ocean and to the more rapid damping of the higher frequencies.

I should like to express my thanks to Dr G. E. R. Deacon of the Admiralty Research Laboratory for suggesting the subject of the present investigation and for his encouragement during the early stages. I am much indebted to Professor H. Jeffreys for many valuable suggestions, and to him and to Dr R. Stoneley for advice in the preparation of this paper. Publication is by kind permission of the Admiralty.

References

Baird, H. F. & Banwell, C. J. 1940 *N.Z. J. Sci. Tech.* **21**B, 314.
Banerji, S. K. 1930 *Phil. Trans.* A, **229**, 287.
Banerji, S. K. 1935 *Proc. Indian Acad. Sci.* **1**, 727.
Bernard, P. 1937 *C.R. Acad. Sci., Paris*, **205**, 163.
Bernard, P. 1941*a* *Bull. Inst. océanogr. Monaco*, **38**, no. 800.
Bernard, P. 1941*b* *Étude sur l'Agitation Microséismique*. Presses Universitaires de France.
Bochner, S. 1932 *Vorlesungen über Fouriersche Integrale*. Leipzig: Akademische Verlagsgesellschaft.
Bradford, D. C. 1935 *Bull. Seism. Soc. Amer.* **25**, 323.
Deacon, G. E. R. 1947 *Nature*, **160**, 419.
Deacon, G. E. R. 1949 *Ann. N.Y. Acad. Sci.* **51**, 3, 475.
Gherzi, E. 1932 *Beitr. Geophys.* **36**, 20.
Gilmore, M. H. 1946 *Bull. Seism. Soc. Amer.* **36**, 89.
Gutenberg, B. 1931 *Bull. Seism. Soc. Amer.* **21**, 1.
Gutenberg, B. 1932 *Handb. Geophys. Berl.* **4**, 263.
Jeffreys, H. 1925 *Proc. Roy. Soc.* A, **107**, 189.
Jeffreys, H. 1926 *Proc. Camb. Phil. Soc.* **23**, 472.
Jeffreys, H. & Jeffreys, B. S. 1946 *Methods of mathematical physics*. Cambridge University Press.
Lamb, H. 1904 *Phil. Trans.* A, **203**, 1.
Lamb, H. 1932 *Hydrodynamics*, 6th ed. Cambridge University Press.
Lee, A. W. 1934 *Geophys. Mem.* no. 62.
Levi-Civita, T. 1925 *Math. Ann.* **93**, 264.
Longuet-Higgins, M. S. & Ursell, F. 1948 *Nature*, **162**, 700.
Miche, M. 1944 *Ann. Ponts Chauss.* **2**, 42.
Pidduck, F. B. 1910 *Proc. Roy. Soc.* A, **83**, 347.
Pidduck, F. B. 1912 *Proc. Roy. Soc.* A, **86**, 396.
Press, F. & Ewing, M. 1948 *Trans. Amer. Geophys. Un.* **29**, 163.
Ramirez, J. E. 1940 *Bull. Seism. Soc. Amer.* **30**, 35–84 and 139–178.
Scholte, J. G. 1943 *Versl. gewone Vergad. Akad. Amst.* **52**, 669.
Sommerfeld, A. 1909 *Ann. Phys., Lpz.*, **28**, 665.
Stoneley, R. 1926 *Mon. Not. R. Astr. Soc. Geophys. Suppl.* **1**, 349.
Tams, E. 1933 *Z. Geophys.* **9**, 23–30 and 295–300.
Whipple, F. J. W. & Lee, A. W. 1935 *Mon. Not. R. Astr. Soc. Geophys. Suppl.* **3**, 287.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES